

Gauge fields and infinite chains of dualities

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Abstract

We show that the particle states of Maxwell's theory, in D dimensions, can be represented in an infinite number of ways by using different gauge fields. Using this result we formulate the dynamics in terms of an infinite set of duality relations which are first order in space-time derivatives. We derive a similar result for the three form in eleven dimensions where such a possibility was first observed in the context of E_{11} . We also give an action formulation for some of the gauge fields. In this paper we give a pedagogical account of the Lorentz and gauge covariant formulation of the irreducible representations of the Poincaré group, used previously in higher spin theories, as this plays a key role in our constructions. It is clear that our results can be generalised to any particle.

1. Introduction

Dirac's wish to treat the electric and magnetic fields of Maxwell's equation in a more symmetric fashion lead him to propose the existence of magnetic monopoles [1]. These were found to occur as regular solutions of spontaneously broken Yang-Mills theories coupled to scalar fields [2]. However, it was Montonen and Olive who proposed that there might be an electromagnetic duality symmetry [3] which was subsequently found to be present in the maximally rigid supersymmetric theory. One of the most important discoveries in supergravity theories was the existence of exceptional symmetries in the maximally supergravity theories [4]. The scalar fields in these theories belong to a coset space constructed from the exceptional symmetry. However, these symmetries generically act on the other fields in the supergravity theory and when acting on the "spin one" fields they act as a kind of electro-magnetic duality symmetry [5].

It has been conjectured that the underlying theory of strings and branes possess a very large Kac-Moody symmetry called E_{11} [6]. This is encoded in a non-linear realisation which possesses an infinite number of fields. The fields are ordered by a level and at low levels the fields in E_{11} are just those of maximal supergravity theory in the dimension being considered. However the E_{11} theory is democratic in that it also contains the dual fields as well as the traditional fields; for example, in eleven dimensions in addition to the graviton and three form, it contains the six form and a field with the index structure $h_{a_1 \dots a_8, b}$ which was the dual of the gravity [6]. Indeed, it was in this paper that an equation of motion in D dimensions of a field with the index structure $h_{a_1 \dots a_{D-3}, b}$ and the usual graviton was given at the linearised level. As this equation was derived from the usual formulation of gravity it was guaranteed to describe gravity in the correct way including the correct degrees of freedom. In fact the equation of motion of the dual graviton had been previously given in five dimensions in reference [7] where it was also pointed out that it had the correct degree of freedom to be gravity. The form of the dual graviton field had also previously been suggested in [8]. Finally, in [9] the action and complete set of gauge symmetries for the dual graviton in arbitrary dimension was constructed along the lines of [6], thereby tying together the results of [6] and [7].

Although, the fields in the E_{11} non-linear realisation are listed for low levels, see for example [10], they are not systematically known at higher levels theory. However, certain results are known, these include all the p -form fields [11,12] in the different dimensions, some of which play a key role in gauged supergravities. Also known are all fields that do not have blocks of ten and eleven indices in eleven dimensions [13]. These fields have a particularly simple form, they are just the usual fields of the maximal supergravity theory, as well as the dual fields just discussed above, as well as an infinite number of fields that consist of adding blocks of 9 indices to these fields. In eleven dimensions these are the fields

$$\{ h_{[1,1]}, A_{[3]}, A_{[6]}, h_{[8,1]}, A_{[9,3]}, A_{[9,6]}, h_{[9,8,1]}, A_{[9,9,3]}, A_{[9,9,6]}, h_{[9,9,8,1]}, \dots \} \quad (1.1)$$

where the numbers in square brackets indicate the number of antisymmetrised indices in each block. It was noted that when decomposed to the little group $SO(9)$ these blocks of nine indices did not transform and so these fields should be just alternative ways of

describing the degrees of freedom given in terms of the three form and graviton. In the sector of the graviton, this conjecture was verified in [14] at the action level. As a result E_{11} encodes an infinite duality symmetry which should be expressed through an infinite series of duality relations which determine the dynamics of the particle. The analogous results for other very extended algebras were given in [15].

In this paper we will show that alternative gauge field representations arise quite generally for any particle. Indeed, they arise naturally from the irreducible unitary representations of the Poincaré group $ISO(1, D - 1)$ that describe any particle moving in Minkowski space-time. These are constructed following the method of Wigner, which involves the induced representation based on the isotropy, or little, group that preserves the momentum in a chosen Lorentz frame [16,17]. In the massless case, this method works with the gauge field and as a result it has a number of ad hoc steps associated with the gauge transformation of this field.

However, there is another type of representation of $ISO(1, D - 1)$ also capable of carrying the massless, irreducible and unitary representations of $ISO(1, D - 1)$, and that is manifestly Lorentz covariant and also gauge invariant [18,19], see also [20,21,22]. This works with the fields strengths and their derivatives rather than the gauge fields. We note also the anterior and different method [23] where field equations for arbitrary gauge fields are also formulated in terms of curvature tensors. When viewed in this way we will show that there is an infinite number of ways of introducing different gauge potentials corresponding to the particular equations one takes to be Bianchi identities and those one considers to be equations of motion. This choice reflects the possibility of the different possible duality transformations one can carry out on the field strengths and all their space-time derivatives. In the case of the spin-2 gauge field, this mechanism was explained in reference [14] using the fields contained in the E_{11} non-linear realisation [6].

In this paper we will carry out this programme for Maxwell's theory in D dimensions and then for the three form in eleven dimensions. As is very well known the latter occurs in the eleven dimensional supergravity theory [4]. We will find the equations of motion for the particles when described in terms of any of the possible gauge fields. We will also find an infinite series of duality relations that encode the dynamics of the particles and involve all the gauge potentials.

For the first sections of the paper we use familiar conventions for writing indices on fields, but as the paper progresses, and the number of different types of indices increases, we use a number of shorthand conventions. We define these as we use them, but for easy reference we give an appendix where these conventions are listed.

2. Spin one and its gauge fields

In this section we illustrate the ideas of this paper in the context of the simplest model, that is, the Maxwell theory in D dimensions. We will show that this system possesses an infinite number of descriptions corresponding to an infinite number of different possible choices of gauge fields. The states of any particle are the irreducible unitary representation of the Poincaré group $ISO(1, D - 1)$ in D dimensions. As we mentioned above, these were first found by Wigner [16] who constructed them as an induced representation of

$ISO(1, D-1)$ with respect to an isotropy subgroup that preserves a fixed momentum and it is this formulation that is most widely known. The representations are labelled by the representations of the isotropy group that they carry. In the massless helicity cases the isotropy subgroup is $SO(D-2)$ and by spin one we mean it carries the vector representation of $SO(D-2)$.

However, for the massless case the Wigner formulation of particle states involves introducing a gauge field and the procedure has a number of ad hoc steps associated with the gauge symmetry of this field. There does exist a much less well known, but equivalent formulation of the Wigner unitary irreducible representations, that is manifestly $SO(1, D-1)$ covariant and, for the massless case, is also manifestly gauge invariant; indeed it involves field strengths and their derivatives and plays an important role in the formulation of nonlinear higher spin theories, see [19,21,22] and refs. therein. We will ask what possible gauge potentials are contained in this representation and in this way we will find an infinite possible choices of gauge potentials. In this section we take the opportunity to give a hopefully very readily understandable account of this formulation of the irreducible representations of $ISO(1, D-1)$ using only knowledge that every physicists knows.

2.1 The Wigner unitary irreducible representation of spin one

As every theorist knows a spin one particle can be described by a rank two field strength $F_{a_1 a_2}$ subject to the Bianchi identity

$$\partial_{[a_1} F_{a_2 a_3]} = 0 \quad (2.1.1)$$

and the equation of motion

$$\partial^a F_{ab} = 0 . \quad (2.1.2)$$

These imply that

$$\partial^{a_1} \partial_{[a_1} F_{a_2 a_3]} = 0 \quad \text{and so} \quad \partial^b \partial_b F_{a_1 a_2} = 0 . \quad (2.1.3)$$

We will hence forth denote $F^{(0)}_{a_1 a_2} := F_{a_1 a_2}$ as this will be the first in a series of objects $F^{(n)}_{a_1 a_2 || b_1 \dots b_n}$ that we will define. We refer to n as the level.

To show that these do indeed describe a spin one we can choose our Lorentz frame so that $k^\mu = (k^+, 0, 0, \dots, 0)$ in light-cone coordinates, whereupon equation (2.1.2) implies that $F^{(0)}_{+a} = 0$ while equation (2.1.1) implies that $k_{[-} F^{(0)}_{ab]} = 0$. Consequently, the only non zero components of the field strength are $F^{(0)}_{-i}$, $i = 1, \dots, D-2$ subject to equation (2.1.3) and these we recognise as the $D-2$ degrees of freedom of a “spin 1”.

We are now going to formulate the above conditions in an alternative manner which will lead to the irreducible unitary representation of $ISO(1, D-1)$ corresponding to spin one, but in such a way that it is manifestly $SO(1, D-1)$ covariant and also gauge invariant. We first observe that the conditions of equation (2.1.1) and (2.1.2) on $F^{(0)}_{a_1 a_2}$ can be rewritten by defining

$$F^{(1)}_{a_1 a_2 || b} := \partial_b F^{(0)}_{a_1 a_2} , \quad (2.1.4)$$

whereupon they are equivalent to the conditions

$$F^{(1)}_{[a_1 a_2 || b]} = 0 = F^{(1)}_{a_1 b ||}{}^b . \quad (2.1.5)$$

In the above and in what follows, we use conventions whereby double bars separate groups of indices that are subject to $GL(D)$ -irreducibility conditions. Thus we recognise the Bianchi identity of equations (2.1.1) as just being the requirement that the tensor $F^{(1)}_{a_1 a_2 || b}$ is $GL(D)$ irreducible. This is the same as stating that $F^{(1)}_{a_1 a_2 || b}$ belongs to the $GL(D)$ Young tableau

$$\begin{array}{|c|c|} \hline a_1 & b \\ \hline a_2 & \\ \hline \end{array} . \quad (2.1.6)$$

The second condition of equation (2.1.2) can be stated as that $F^{(1)}_{a_1 a_2 || b}$ is also a $SO(1, D-1)$ irreducible tensor. A Young tableau can be of $GL(D)$ or $SO(1, D-1)$ type. The former encodes constraints that involve the antisymmetrisation, or symmetrisation, of certain groups of indices, such as in the first of the equation in (2.1.5), however, the latter tableau also encodes trace conditions, such as in the second equations in (2.1.5). As a result, $F^{(1)}_{a_1 a_2 || b}$ belongs to the irreducible representation associated with the $SO(1, D-1)$ Young tableau given above in (2.1.6). The conditions encoded in the Young tableau are just those required to give an irreducible representation of the relevant group. A discussion a Young tableaux can be found in [25,26].

We now take another derivative and consider the quantity

$$F^{(2)}_{a_1 a_2 || b_1 b_2} := \partial_{b_2} F^{(1)}_{a_1 a_2 || b_1} , \quad (2.1.7)$$

which satisfies the conditions

$$F^{(2)}_{[a_1 a_2 || b_1] b_2} = 0 = F^{(2)}_{a_1 b || b_2}{}^b , \quad F^{(2)}_{a_1 a_2 || [b_1 b_2]} = 0 = F^{(2)}_{a_1 a_2 || b}{}^b . \quad (2.1.8)$$

The first two conditions are obvious from the definition of equation (2.1.7) and equation (2.1.5) while the last two conditions follow by substituting equation (2.1.4) into equation (2.1.7) and using equation (2.1.3). By considering $\partial^{b_1} \partial_{[b_1} F^{(1)}_{b_2 b_3 || b_4]}$, it follows from the constraints of equations (2.1.5) and (2.1.8) that

$$\partial_c \partial^c F^{(1)}_{a_1 a_2 || b} = 0 = \partial_c \partial^c F^{(2)}_{a_1 a_2 || b_1 b_2} . \quad (2.1.9)$$

The constraints of equation (2.1.8) are equivalent to demanding that $F^{(2)}_{a_1 a_2 || b_1 b_2}$ has the properties associated with the $SO(1, D-1)$ Young tableau given by

$$F^{(2)}_{a_1 a_2 || b_1 b_2} \sim \begin{array}{|c|c|c|} \hline a_1 & b_1 & b_2 \\ \hline a_2 & & \\ \hline \end{array} . \quad (2.1.10)$$

We now generalise the above to higher levels and define a sequence of objects up to level n :

$$\{F^{(p)}_{a_1 a_2 || b_1 \dots b_p}\} , \quad p = 0, 1, \dots, n , \quad (2.1.11)$$

where we assume that

$$F^{(p)}_{[a_1 a_2 || b_1] \dots b_p} = 0 = F^{(p)}_{a_1 b || b_2 \dots b_p}{}^b , \quad F^{(p)}_{a_1 a_2 || b_1 \dots b_{p-2} c}{}^c = 0 , \quad p = 0, 1, \dots, n , \quad (2.1.12)$$

and

$$F^{(p)}_{a_1 a_2 \| b_1 \dots b_p} = F^{(p)}_{a_1 a_2 \| (b_1 \dots b_p)}, \quad p = 0, 1, \dots, n. \quad (2.1.13)$$

Proceeding to the next level $n + 1$ we define

$$F^{(n+1)}_{a_1 a_2 \| b_1 \dots b_{n+1}} := \partial_{b_{n+1}} F^{(n)}_{a_1 a_2 \| b_1 \dots b_n}. \quad (2.1.14)$$

It is now straightforward to show that $F^{(n+1)}_{a_1 a_2 \| b_1 \dots b_{n+1}}$ obeys equations (2.1.12) and (2.1.13) but with $p = n + 1$. Thus by induction we have an infinite set of objects which obey the constraints of equations (2.1.12) and (2.1.13) for all p .

To summarise, one has a description of “spin one” in D dimensions in terms of an infinite the set of objects

$$\mathcal{W} = \{F^{(n)}_{a_1 a_2 \| b_1 \dots b_n} \sim \begin{array}{|c|c|c|c|} \hline a_1 & b_1 & \dots & b_n \\ \hline a_2 & & & \\ \hline \end{array}, \quad n = 0, 1, \dots\} \quad (2.1.15)$$

which are related by equation (2.1.14) and which are subject to the constraints that are encoded in the $SO(1, D - 1)$ Young tableau.

The discussion above of all the higher level objects may seem at first sight as a bit redundant, but it has an important interpretation. The objects of equation (2.1.15) carry Wigner’s unitary irreducible representation of $ISO(1, D - 1)$ which corresponds to “spin one”. Indeed, there exists a map from Wigner’s unitary irreducible representation of $ISO(1, D - 1)$ for “spin one” where all states are labelled by momentum and polarisation tensors, to \mathcal{W} , where the states are labelled by Lorentz tensors. The action of the Lorentz generators is as usual while the translations acts as

$$P_c(F^{(n)}_{a_1 a_2 \| b_1 \dots b_n}) = F^{(n+1)}_{a_1 a_2 \| b_1 \dots b_n c}. \quad (2.1.16)$$

The reader may verify that \mathcal{W} does indeed carry a representation of $ISO(1, D - 1)$. We note that this representation is not irreducible, as it contains infinitely many ideals \mathcal{W}_{n_0} , namely the modules obtained by truncating the level n to any minimum value n_0 . As we shall see below, it is nevertheless possible to reconstruct \mathcal{W} from any ideal \mathcal{W}_{n_0} by integration with suitable boundary conditions imposed, conditions that we shall leave unspecified below for the sake of simplicity.

As we have mentioned, the advantage of using the above representation \mathcal{W} and its generalisations to particles of other spin, is that it is manifestly Lorentz covariant and in the massless case also gauge invariant and so it does not require a particular representation in terms of a gauge potential and its associated gauge transformations. This will prove key in what follows. The representations \mathcal{W} , and its generalisations are equivalent to the formulation of these representations given by the Wigner method of induced representations [16,17].

The discussion above was pedagogical but to some extent a simplified account using just ideas that are universally known. In fact, the procedure is best understood from a slightly different and more abstract viewpoint. We should start from the beginning with the

fully indecomposable $ISO(1, D-1)$ representation of equation (2.1.15), the fields of which by definition are subject to the $SO(1, D-1)$ conditions encoded in the Young tableaux, and related by the derivative condition of equation (2.1.14). As should be the case for this representation, these equations imply the on-shell dynamics. This should be apparent from the above, for example the constraints on $F^{(1)}_{a_1 a_2 || b}$ and the fact that $\partial_b F^{(1)}_{a_1 a_2} = F^{(1)}_{a_1 a_2 || b}$ implies the usual Bianchi and equation of motion for a spin one particle. The above Lorentz-covariant method of describing the particle states is an example of what is called the *unfolded* description of field theory dynamics and was initiated by M. Vasiliev, see [19] and references therein. This is a formulation of the dynamics by a set of first order differential equations; in this case equations (2.1.14) for all n together with the constraints just discussed. As we said, unfolded formulation plays the central role in nonlinear higher spin gravity theories.

The representation \mathcal{W} of equation (2.1.15) contains the field $F^{(0)}_{a_1 a_2}$ and all its derivatives and one can think of this as the field and all its derivatives at a given space-time point. Using Taylor's theorem, we then know the fields at all space-time points as the coefficients in the expansion are the just mentioned quantities.

It is clear from the above construction that if we have the representation \mathcal{W} up to level n then we can, by acting with space-time derivatives, construct all the higher level elements in the representation; indeed this is what we did above. However, it is also possible to reconstruct \mathcal{W} if we have all the elements at, and above, any given level n , which we denoted \mathcal{W}_n above. The reconstruction is possible by using the Poincaré lemma. The fact that integration is required is to be expected, as \mathcal{W} is a fully indecomposable representation. Let us consider $F^{(p)}_{a_1 a_2 || b_1 \dots b_p}$, $p \geq n$, which is subject to all the constraints dictated by its $SO(1, D-1)$ Young tableau of equation (2.1.15). These, in particular, imply that

$$\partial_{[b_{n+1}]} F^{(n)}_{a_1 a_2 || b_1 \dots [b_n]} = 0. \quad (2.1.17)$$

As a result one can deduce, using the usual Poincaré lemma, that there exists an object $F^{(n-1)}_{a_1 a_2 || b_1 \dots b_{n-1}}$ such that

$$F^{(n)}_{a_1 a_2 || b_1 \dots b_n} = \partial_{b_n} F^{(n-1)}_{a_1 a_2 || b_1 \dots b_{n-1}}. \quad (2.1.18)$$

From the fact that $F^{(n)}_{a_1 a_2 || b_1 \dots b_n}$ satisfies the algebraic constraints associated with its $SO(1, D-1)$ Young tableau, it follows that $F^{(n-1)}_{a_1 a_2 || b_1 \dots b_{n-1}}$ obeys the analogous constraints. Proceeding in this way we reconstruct the representation down to level zero.

It is instructive to find the degrees of freedom contained in the higher level elements of the representation \mathcal{W} . Let us consider $F^{(1)}_{a_1 a_2 || b}$ which is subject to the constraints of equation (2.1.8) in conjunction with equation (2.1.7), but not its connection to level zero, that is, to $F^{(1)}_{a_1 a_2 || b}$ of equation (2.1.4). These differential constraints imply that $F^{(1)}_{a_1 a_2 || b}$ is divergenceless and curl-free on its two sets of indices, and as a result $F^{(1)}_{a_1 a_2 || b}$ is harmonic. One goes to momentum space and takes $k^\mu = (k^+, 0, 0, \dots, 0)$, as before, and finds that the last three equations imply that the indices a_1, a_2 and b cannot take the value $+$. The curl-free equations then imply that the only non-zero components are $F^{(1)}_{-i || -}$, $i = 1, \dots, D-2$.

Hence we find it contains the required $D - 2$ degrees of freedom. A similar analysis at level n implies that the only non-zero components of $F^{(n)}_{a_1 a_2 \parallel b_1 \dots b_n}$ are $F^{(n)}_{-i \parallel - \dots -}$. We note that, in the chosen Lorentz frame, all the non-vanishing components are related by $F^{(n)}_{-i \parallel - \dots -} = k_- \dots k_- F^{(0)}_{-i}$ and reproduce all the Taylor coefficients of an on-shell Maxwell field at any given point, therefore allowing to reconstruct the field in the neighbourhood of that point.

2.2 Dualities and Gauge potentials

The representation \mathcal{W} of $ISO(1, D - 1)$ describes the states of a spin one in a way that is manifestly gauge invariant, since it is constructed from field strengths and their derivatives. We now consider what gauge potentials are implied by this representation. We begin at the lowest level. Every theorist knows that the Bianchi identity of equation (2.1.1) can be solved in terms of a gauge potential $A_a^{(0)}$ as

$$F^{(0)}_{a_1 a_2} = 2 \partial_{[a_1} A^{(0)}_{a_2]} , \quad (2.2.1)$$

with the usual gauge symmetry $\delta A_a = \partial_a \Lambda$.

However, we are free to choose which of the equations in the representation we would like to solve and we can equally well choose to solve equation (2.1.2) even though this is usually thought of as the equation of motion. To achieve this we define

$$G_{a_1 \dots a_{D-2}} = \frac{1}{2} \epsilon_{a_1 \dots a_{D-2}}{}^{b_1 b_2} F_{b_1 b_2} , \quad (2.2.2)$$

whereupon the equation (2.1.2) becomes

$$\epsilon^{a_1 a_2 b_1 \dots b_{D-2}} \partial_{a_2} G_{b_1 \dots b_{D-2}} = 0 , \quad (2.2.3)$$

with the solution

$$G_{b_1 \dots b_{D-2}} = \partial_{[b_1} A^{(0)}_{b_2 \dots b_{D-2}]} , \quad (2.2.4)$$

that is, in terms of a gauge field $A^{(0)}_{b_1 \dots b_{D-3}}$ with the gauge symmetry

$$\delta_\lambda A^{(0)}_{b_1 \dots b_{D-3}} = \partial_{[b_1} \lambda_{b_2 \dots b_{D-3}]} .$$

However as we can reconstruct \mathcal{W} from \mathcal{W}_n by integration, we choose to carry out a duality at level n rather than those at level zero. Let us first consider level one and define

$$G^{(1)}_{c_1 \dots c_{D-1} \parallel a_1 a_2} := \epsilon_{c_1 \dots c_{D-1} b} F^{(1)}_{a_1 a_2 \parallel}{}^b . \quad (2.2.5)$$

It is straightforward to verify, using equation (2.1.2), that is, the trace constraint $F^{(1)}_{ab \parallel}{}^b = \partial^b F_{ab} = 0$, that

$$G^{(1)}_{[c_1 \dots c_{D-1} \parallel a_1] a_2} = 0 , \quad (2.2.6)$$

Thus, $G^{(1)}_{c_1 \dots c_{D-1} \| a_1 a_2}$ sits inside the irreducible representation of $GL(D)$ that transforms according to the following $GL(D)$ Young tableau

$$G^{(1)}_{c_1 \dots c_{D-1} \| a_1 a_2} \sim \begin{array}{|c|c|} \hline c_1 & a_1 \\ \hline c_2 & a_2 \\ \hline \vdots & \\ \hline c_{D-1} & \\ \hline \end{array} . \quad (2.2.7)$$

However, it is easy to see that $G^{(1)}_{c_1 \dots c_{D-2} b \| a_2} \neq 0$ and so the constraints on this object are not those of an $SO(1, D-1)$ Young tableau, which are single traceless by definition. It does however satisfy a higher order trace condition, and using the first equation of (2.1.5), one finds that

$$G^{(1)}_{a_1 a_2 c_1 \dots c_{D-3} \| a_1 a_2} = 0 . \quad (2.2.8)$$

We can think of equation (2.2.8) as the equation of motion and equation (2.2.6) as the Bianchi identity for the particle when written in terms of $G^{(1)}_{c_1 \dots c_{D-1} \| a_1 a_2}$. Indeed we will show below that equation (2.2.8) follows from extremising an action. We note the usual interchange of equation of motion and Bianchi identity under a duality transformation.

We next consider what derivative constraints $G^{(1)}_{c_1 \dots c_{D-1} \| a_1 a_2}$ satisfies. Using equation (2.1.8) and the definition (2.2.5) we find that

$$\partial_{[a_1} G^{(1)}_{c_1 \dots c_{D-1} \| a_2 a_3]} = 0 = \partial_{[c_1} G^{(1)}_{c_2 \dots c_{D-1} \| a_1 a_2]} , \quad (2.2.9)$$

and

$$\partial^e G^{(1)}_{c_1 \dots c_{D-1} \| e a} = 0 = \partial^d G^{(1)}_{d c_2 \dots c_{D-1} \| a_1 a_2} . \quad (2.2.10)$$

Equations (2.2.9) can be summarised by defining

$$G^{(2)}_{c_1 \dots c_{D-1} \| a_1 a_2 \| b} := \partial_b G^{(1)}_{c_1 \dots c_{D-1} \| a_2 a_3} , \quad (2.2.11)$$

and demanding that it belong to the $GL(D)$ Young tableau

$$\begin{array}{|c|c|c|} \hline c_1 & a_1 & b \\ \hline c_2 & a_2 & \\ \hline \vdots & & \\ \hline c_{D-1} & & \\ \hline \end{array} . \quad (2.2.12)$$

The tensor $G^{(2)}_{c_1 \dots c_{D-1} \| a_1 a_2 \| b}$ satisfies trace conditions inherited from equation (2.2.10) and an obvious second order trace condition inherited from (2.2.8). Because of the latter condition, these are not the single-trace conditions associated with an $SO(1, D-1)$ -irreducible Young tableau.

Before introducing potentials, we can proceed a but further as we did for $F_{a_1 a_2}$ to construct an infinite dimensional representation based on $G_{c_1 \dots c_{D-1} \| a_1 a_2}^{(1)}$ by defining

$$G_{c_1 \dots c_{D-1} \| a_1 a_2 \| b_1 \dots b_{n+1}}^{(n+1)} := \partial_{b_{n+1}} G_{c_1 \dots c_{D-1} \| a_2 a_3 \| b_1 \dots b_n}^{(n)} , \quad (2.2.13)$$

and find the constraints that the new objects satisfies. However, we will not pursue this further in this work.

We now choose to regard equations (2.2.9) as Bianchi identities and so solve these instead of the Bianchi identity at level zero. Using the generalised Poincaré lemma spelled out in section 5 of [28], we find that

$$G_{c_1 \dots c_{D-1} \| a_2 a_3}^{(1)} = \partial^{[a_1} \partial_{[c_1} A_{c_2 \dots c_{D-1} \| a_2]}^{(1)} . \quad (2.2.14)$$

Thus we find a description in terms of a gauge field $A_{a_1 \dots a_{D-2} \| b}$ which satisfies the $GL(D)$ irreducibility condition

$$A_{[a_1 \dots a_{D-2} \| b]}^{(1)} = 0 . \quad (2.2.15)$$

The expression for the field strength in terms of the gauge field can be written in the form of a Young tableau as follows

$$\begin{array}{|c|c|} \hline \partial_{c_1} & \partial_{a_1} \\ \hline c_2 & a_2 \\ \hline \vdots & \\ \hline c_{D-1} & \\ \hline \end{array} . \quad (2.2.16)$$

The fields strength $G_{c_1 \dots c_{D-1} \| a_2 a_3}^{(1)}$ is invariant under the following gauge transformations featuring two independent $GL(D)$ -irreducible gauge parameters $\lambda_{a_1 a_2 \dots a_{D-3} \| b}^{(1)}$ and $\lambda_{a_1 a_2 \dots a_{D-2}}^{(2)}$:

$$\delta_\lambda A_{a_1 \dots a_{D-2} \| b}^{(1)} = (D-2) \partial_{[a_1} \lambda_{a_2 \dots a_{D-2} \| b]}^{(1)} + \partial_b \lambda_{a_1 \dots a_{D-2}}^{(2)} + (-1)^{D-1} \partial_{[a_1} \lambda_{a_2 \dots a_{D-2}] b}^{(2)} . \quad (2.2.17)$$

These two transformations can be represented by the tableaux

$$\begin{array}{|c|c|} \hline a_1 & b \\ \hline a_2 & \\ \hline \vdots & \\ \hline \partial_{a_{D-2}} & \\ \hline \end{array} , \quad \begin{array}{|c|c|} \hline a_1 & \partial_b \\ \hline a_2 & \\ \hline \vdots & \\ \hline a_{D-2} & \\ \hline \end{array} . \quad (2.2.18)$$

We now turn to the case of dualising the object in the irreducible representation of $SO(1, D-1)$ given in equation (2.1.15) at level n . To this end, we need a more

streamlined index notation. We denote $A_{a[n]} \equiv A_{[a_1 \dots a_n]} \equiv A_{a_1 \dots a_n}$ and similarly for all blocks of antisymmetric indices. Similarly, for groups of symmetric indices, we use $S_{a(n)} \equiv S_{(a_1 \dots a_n)} \equiv S_{a_1 \dots a_n}$, with strength-one (anti)symmetrisation convention. Using this notation we define

$$G^{(n)}_{c^1[D-1] \parallel \dots \parallel c^n[D-1] \parallel a_1 a_2} := \epsilon_{c^1[D-1]e_1} \dots \epsilon_{c^n[D-1]e_n} F^{(n)}_{a_1 a_2}{}^{e(n)}. \quad (2.2.19)$$

Using equations (2.1.12) and (2.1.13) one can show that $G^{(n)}_{c^1[D-1] \parallel \dots \parallel c^n[D-1] \parallel a_1 a_2}$ obeys the following over-antisymmetrisation constraints

$$G^{(n)}_{c^1[D-1] \parallel \dots \parallel c_D^1 c^i[D-2] \parallel \dots \parallel c^n[D-1] \parallel a_1 a_2} = 0 = G^{(n)}_{c^1[D-1] \parallel \dots \parallel c^j[D-1] \parallel \dots \parallel c^n[D-1] \parallel c_D^j a_2}, \quad (2.2.20)$$

$$i \in \{2, \dots, n\}, \quad j \in \{1, 2, \dots, n\}.$$

As a result, $G^{(n)}$ belongs to the $GL(D)$ Young tableau

c_1^1	\dots	c_1^n	a_1
c_2^1	\dots	c_2^n	a_2
\vdots	\dots	\vdots	
c_{D-1}^1	\dots	c_{D-1}^n	

(2.2.21)

It is straightforward to show that although $G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel a_1 a_2}$ does not satisfy any single trace conditions it does satisfy a double and a $(D-1)$ -trace condition which are given by

$$G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-3] a_1 a_2}{}^{a_1 a_2} = 0 = G^{(n)}_{c[D-1] \parallel \dots \parallel c^{[D-1]} \parallel a_1 a_2}. \quad (2.2.22)$$

The dynamics of the “spin one” when expressed in terms of the field strength $G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel a_1 a_2}$ is given by equations (2.2.20) and (2.2.22) which replace equations (2.1.1) and (2.1.2) of the usual formulation in terms of the field strength $F_{a_1 a_2}^{(0)}$. We can think of equation (2.2.20) as generalised Bianchi identities at level $n-1$ and equations (2.2.22), which involve traces, as equations of motion.

Using the equations (2.2.19) and equation (2.1.13), we find that the field strength $G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel a_1 a_2}$ also obeys the curl-free conditions

$$\partial_c G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel a_1 a_2} = 0 = \partial_{a_1} G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel a_2 a_3}, \quad (2.2.23)$$

as well as the divergence-free conditions

$$\partial^e G^{(n)}_{ec[D-2] \parallel \dots \parallel d[D-1] \parallel a_1 a_2} = 0 = \partial^e G^{(n)}_{c[D-1] \parallel \dots \parallel d[D-1] \parallel ea}. \quad (2.2.24)$$

We note that (2.2.23) will be the generalised Bianchi identities at the level n while the equation of motion will not be (2.2.24) but instead the higher-trace constraints (2.2.20).

We find the gauge potential at level n by applying the generalised Poincaré lemma [28] to equation (2.2.23). The result is

$$G^{(n)}_{c^1[D-1]]\dots\|c^n[D-1]]^{a_1 a_2} = \partial^{[a_1} \partial_{[c_1^1} \dots \partial_{c_1^n} A^{(n)}_{c^1[D-2]]\dots\|c^n[D-2]]^{a_2]} , \quad (2.2.25)$$

where the gauge potential $A^{(n)}_{c[D-2]]\dots\|d[D-2]]_a$ is an irreducible tensor of $GL(D)$ and so obeys the constraints

$$A^{(n)}_{c[D-2]]\dots\|[f[D-2]]_a] = 0 = A^{(n)}_{[c[D-2]]\dots\|[f_1]|f[D-3]]_a} . \quad (2.2.27)$$

As a result $A^{(n)}_{c[D-2]]\dots\|[f[D-2]]_a}$ belongs to the $GL(D)$ Young tableau

c_1	\dots	f_1	a
c_2	\dots	f_2	
\vdots	\dots	\vdots	
c_{D-2}	\dots	f_{D-2}	

(2.2.28)

The potential has no trace constraint and has gauge symmetries involving two gauge parameters, $\Lambda^{(n,1)}_{[D-2,\dots,D-2]}$ and $\Lambda^{(n,2)}_{[D-2,\dots,D-2,D-3,1]}$.

We note that the field strength $G^{(n)}_{c[D-1]]\dots\|d[D-1]]_{a_1 a_2}$ involves $n + 1$ space-time derivatives, instead of the more familiar two derivatives. The expression of the field strength $G^{(n)}_{c[D-1]]\dots\|[f[D-1]]_{a_1 a_2}$ in terms of the gauge field $A^{(n)}_{c[D-2]]\dots\|[f[D-2]]_a}$ given in equation (2.2.25) can be expressed in Young tableau language as

∂_{c_1}	\dots	∂_{f_1}	∂_{a_1}
c_2	\dots	f_2	a_2
\vdots	\dots	\vdots	
c_{D-2}	\dots	f_{D-2}	
c_{D-1}	\dots	f_{D-1}	

(2.2.28)

The fields strength $G^{(n)}_{c[D-1]]\dots\|d[D-1]]_{a_1 a_2}$ when expressed in terms of its gauge potential in equation (2.2.25) automatically obeys the Bianchi identities of equations (2.2.20) and (2.2.23). However, it also obeys equation (2.2.22) which is the equation of motion for the gauge field, and we stress that it contains $n + 1$ space-time derivatives. The presence of higher space-time derivatives is characteristic of the equations of motion of higher spin fields and indeed of any mixed symmetry fields, when formulated in terms of curvatures. For a recent discussion and references along those lines, see [27].

Thus we have found that there is an infinite number of ways of representing the particle states of “spin one” corresponding to the existence of an infinite number of different possible gauge potentials arising from the infinite number of ways of dualising the field strength and its descendants that occur in the Lorentz-covariant unfolded module \mathcal{W} . We repeat that the latter module carries the irreducible unitary representation of $ISO(1, D-1)$ that describes the states of the “spin one”.

We now turn to a key point of this paper, which is the duality relations between the first-order derivatives of the potentials. To obtain these, we will first obtain duality relation between the $(n+1)$ -derivative field strengths when they are expressed in terms of their respective gauge potentials. We begin at level zero and in particular equation (2.2.2) which now relates the gauge field A_a to the gauge field $A_{a[D-3]}$. This duality relation is of a familiar type in so much as it relates equations of motion to Bianchi identities. However, once we have substituted in the gauge potentials the Bianchi identities hold automatically and so the relations imply the equations of motion.

We now consider duality relation at level $n=1$, which was given in equation (2.2.5) and that can be written as

$$\partial_{[a_1} \partial_{[c_1} A^{(1)}_{c[D-2]]a_2] = \epsilon_{c[D-1]b} \partial^b \partial_{[a_1} A^{(0)}_{a_2]} . \quad (2.2.29)$$

We note that the Bianchi identities of equations of (2.1.1), or equivalently the first equation in (2.1.5), and equation (2.2.6) are automatically satisfied. However as the duality relation interchange Bianchi identities with equations of motion for the two fields we find that equation (2.2.29) automatically imposes the equations of motion of the two fields, namely, the second equation of (2.1.5) and equation (2.2.8), *i.e.*

$$\partial^b F_{ab} = 0 \quad (2.2.30)$$

and

$$\partial^{a_1} \partial_{[a_1} A^{(1)}_{a_2 c[D-3]]}{}^{a_2} = 0 . \quad (2.2.31)$$

Another, more direct way of getting these two equations directly from (2.2.29) is to antisymmetrise all the c indices together with a_1 of that equation, which gives (2.2.30), or to take its double trace, which gives of (2.2.31).

We now consider the duality relations at higher levels. We begin with the relation (2.2.19) but write it in the form

$$\begin{aligned} G^{(n)}_{c[D-1] \dots b[D-1] a_1 a_2} &= \epsilon_{b[D-1]}^f G^{(n-1)}_{c[D-1] \dots a_1 a_2} f = \\ &= \epsilon_{b[D-1]}^f \partial_f G^{(n-1)}_{c[D-1] \dots a_1 a_2} , \end{aligned} \quad (2.2.32)$$

which relates field strengths at adjacent levels. We now examine the effect of imposing the Bianchi identities and equations of motion on each of these field strengths without assuming that they are given in terms of the gauge fields. For indices that are not involved in the duality, that is do not occur on the epsilon symbol, the constraints on one side of the equation obviously hold on the other side. As a result we now consider the constraints that

involve indices that occur in the duality. The Bianchi identities of $G^{(n)}_{c[D-1]||...||b[D-1]||a_1a_2}$ of equation (2.2.20) imply the trace conditions for $G^{(n-1)}_{c[D-1]||...||a_1a_2||f}$, namely

$$G^{(n)}_{c[D-1]||...||b[D-1]||a_1a_2} = 0 \iff G^{(n-1)}_{c[D-1]||...||a_1b||}{}^b = 0, \quad (2.2.33)$$

and

$$G^{(n)}_{[c[D-1]||...||b_1]b[D-2]||a_1a_2} = 0 \iff G^{(n-1)}_{ec[D-2]||...||a_1a_2||}{}^e = 0. \quad (2.2.34)$$

Conversely the Bianchi identities of $G^{(n-1)}_{c[D-1]||...||a_1a_2||f}$ imply the trace conditions of $G^{(n)}_{c[D-1]||...||b[D-1]||a_1a_2}$ of equation (2.2.22), namely

$$G^{(n-1)}_{[c[D-1]||...||a_1a_2||f]} = 0 \iff G^{(n)}_{c[D-1]||...||}{}^{c[D-1]}_{||a_1a_2} = 0, \quad (2.2.35)$$

and

$$G^{(n-1)}_{c[D-1]||...||a_1a_2||f]} = 0 \iff G^{(n)}_{c[D-1]||...||a_1a_2b[D-3]||}{}^{a_1a_2} = 0. \quad (2.2.36)$$

Substituting for the gauge field in the duality relation of equation (2.2.32) yields

$$\partial^{[a_1}\partial_{[c_1}\dots\partial_{b_1}A^{(n)}_{b[D-2]||...||c[D-2]||}{}^{a_2]} = \epsilon_{b[D-1]}{}^f\partial_f\partial^{[a_1}\partial_{[c_1}\dots A^{(n-1)}_{...||c[D-2]||}{}^{a_2]}. \quad (2.2.37)$$

Once we have substituted the gauge fields in the field strengths, the Bianchi identities, which occur on the left hand-sides of equations (2.2.33-36), are automatically satisfied and as a result the trace conditions on the dual field strengths are now enforced. In particular, examining equation (2.2.35) and (2.2.36), we now find that their left-hand sides vanish automatically and so the gauge field $A^{(n-1)}_{[D-1]||...||a}$ does not appear in this relation. Consequently, the right-hand side of these relations are enforced and we find that the field strength $G^{(n)}_{c[D-1]||...||b[D-1]||a_1a_2}$ satisfy the trace conditions, which are the equations of motion for the gauge field $A^{(n)}_{c[D-2]||...||b[D-1]||a}$.

We note that the field strength is symmetric under the exchanges of its columns of $D-1$ indices and so the trace conditions hold on all these columns and not just for the ones displayed above. Hence the duality condition of equation (2.2.37) implies the equation of motion for the “spin one” in the formulation with the level n gauge field. Examining equations (2.2.33) and (2.2.34) we find a similar conclusion but now the gauge field $A^{(n)}_{c[D-2]||...b[D-1]||a}$ is eliminated and we have the equation of motion for the gauge field $A^{(n-1)}_{c[D-2]||...b[D-2]||a}$ field.

Equations (2.2.37) can be thought of as an infinite set of duality relations for $n = 1, 2, \dots$, the first of which is given in equation (2.2.29). We note that they involve ever increasing numbers of space-time derivatives as n increases. However, as we now show we can integrate these equations such that they only involve a single space-time derivative. At the lowest level we find, integrating equation (2.2.29), that

$$\partial_c A^{(1)}_{c[D-2]||a} = \epsilon_{c[D-1]b} \partial^b A^{(0)}_a + \partial_a \Xi_{c[D-1]}, \quad (2.2.38)$$

where the last term is the general solution of the homogeneous equation. We can rewrite equation (2.2.38) as

$$\partial_{c_1} A^{(1)}_{c[D-2]||a} = \epsilon_{c[D-1]b} \{ \partial^b A^{(0)}_a + \partial_a \Xi^b \}, \quad (2.2.39)$$

where $\Xi_{c[D-1]} = \epsilon_{c[D-1]b} \Xi^b$. However, since Ξ^b is arbitrary, we can shift it as $\Xi^b \rightarrow \Xi^b - A^b$ whereupon our original equation becomes

$$\partial_c A^{(1)}_{c[D-2]||a} = \epsilon_{c[D-1]}^b \partial_{[b} A^{(0)}_{a]} + \partial_a \Xi_{c[D-1]}, \quad (2.2.40)$$

We observe that the equation is now invariant under the gauge transformations of the original gauge field $A^{(0)}_a$. Antisymmetrising on $\{c_1, \dots, c_{D-1}, a\}$, we find that $\partial_c \Xi_{c[D-1]} = 0$ and so $\Xi_{c[D-1]} = \partial_c \Xi_{c[D-2]}$. Substituting this back in equation (2.2.38) it becomes

$$\partial_{c_1} A^{(1)}_{c[D-2]||a} = \epsilon_{c[D-1]}^b 2\partial_{[b} A^{(0)}_{a]} + \partial_a \partial_c \Xi_{c[D-2]}. \quad (2.2.41)$$

We recognise that the presence of the last term ensures the invariance of (2.2.41) under the gauge transformation of the second type in equation (2.2.18), which acts as a shift symmetry on $\Xi_{c[D-2]}$. The price for the integration is that the equation is now gauge-invariant only at the price of an extra field with a shift symmetry. To eliminate the extra field requires that we differentiate and antisymmetrise with the a index, so recovering the original relation of equation (2.2.29).

Integrating at higher levels, in particular equation (2.2.37), we find that

$$\begin{aligned} \partial_{[b_1} A^{(n)}_{b[D-2]||c^1[D-2]||\dots||c^{n-1}[D-2]||a} &= \epsilon_{b[D-1]f} \partial^f A^{(n-1)}_{c^1[D-2]||\dots||c^{n-1}[D-2]||a} + \\ &+ Y \left(\partial_a \Sigma_{b[D-1]||c^1[D-2]||\dots||c^{n-1}[D-2]} + \partial_{c^{n-1}} \Xi_{b[D-1]||c^1[D-2]||\dots||c^{n-1}[D-3]||a} \right), \\ n &= 0, 1, 2, \dots, \end{aligned} \quad (2.2.42)$$

where $Y(\cdot)$ denotes the projection on the $GL(D)$ Young tableau with index structure $\{c^1[D-1]||\dots||c^{n-1}[D-1]||a\}$. Using arguments similar to those given below equation (2.2.38) one can bring the duality relation to a form that is invariant under certain of the gauge transformations of the field $A^{(n-1)}_{c^1[D-2]||\dots||c_{n-1}[D-2]||a}$ and it then holds modulo the remaining gauge transformations of the two fields,

Rather than constructing the infinite set of duality relations beginning with the gauge field $A_a^{(0)}$ we can alternatively use the level zero gauge field $A_{b_1\dots b_{D-3}}^{(0)}$ and repeat all the above steps. Including this step we find a formulation of the “spin one” field in terms of the following gauge fields

$$\begin{aligned} &A_{[1]}, A_{[D-3]}, A_{[D-2,1]}, A_{[D-2,D-3]}, A_{[D-2,D-2,1]}, \\ &A_{[D-2,D-2,D-3]}, \dots, A_{[D-2,\dots,D-2,1]}, A_{[D-2,\dots,D-2,D-3]}, \dots \end{aligned} \quad (2.2.43)$$

where the numbers shown as subscripts between square brackets indicate the number of indices in each block, that is the length of columns in the corresponding Young tableau.

Thus, in summary we have shown that the spin one can be described by an infinite set of duality equations which are first order in space-time derivatives but only hold modulo certain gauge transformations, One might suspect that these duality relations are invariant under an infinite duality symmetry, modulo the gauge transformations. Indeed one might suppose that this can be formulated as a non-linear realisation of an algebra with generators that carry the same indices that are those carried by the gauge fields but raised.

2.3 Action principle for the dual potential $A_{c[D-2]||a}^{(1)}$

In this section, we follow the lines sketched in [14] and give the action describing the dynamics of a Maxwell field in terms of the dual potential $A_{c[D-2]||a}^{(1)}$ introduced above and sometimes denoted $A_{[D-2,1]}$, for the sake of brevity. The way we recover the dynamics (2.2.8) is interestingly subtle. As explained in the context of the Fierz–Pauli theory in [14], the various dual actions involve more and more off-shell fields and are therefore less and less economical. The special case of spin-1 is simpler but allows us to see in a very explicit way the mechanism whereby the extra off-shell fields disappear from the dynamics on shell.

We start, as it should, with the Maxwell action, and integrate by part:

$$S[A] = -\frac{1}{2} \int d^D x \partial_a A_b (\partial^a A^b - \partial^b A^a) = -\frac{1}{2} \int d^D x (\partial_a A_b \partial^a A^b - \partial_a A^a \partial_b A^b), \quad (2.3.1)$$

dropping the boundary term. Introducing the following parent action

$$S[Y, P] = \int d^D x (P_a|{}^b \partial_c Y^{ca}|_b - \frac{1}{2} P^a|{}_b P^b|{}_a + \frac{1}{2} P^a|{}_a P^b|{}_b) \quad (2.3.2)$$

that features two fields, $Y^{ca}|_b = -Y^{ac}|_b$ and $P_a|{}^b$, we reproduce the original action (2.3.1) upon extremising $S[Y, P]$ with respect to Y :

$$\partial_{[c} P_{a]}|{}^b = 0 \quad \Leftrightarrow \quad P_a|{}^b = \partial_a A^b, \quad (2.3.3)$$

and plugging back into (2.3.2). On the other hand, $P_a|{}^b$ is an auxiliary field, so that extremising the action with respect to it enables one to express it in terms of the Y field:

$$P_b|{}^a = \partial_c Y^{ca}|_b - \frac{1}{(D-1)} \delta_b^a \partial_c Y^{cd}|_d. \quad (2.3.4)$$

Plugging that expression for P inside the parent action (2.3.2) yields the action

$$S[Y^{ca}|_b] = \int d^D x \left(\frac{1}{2} \partial_c Y^{ca}|_b \partial^d Y_{da}|{}^b - \frac{1}{2(D-1)} \partial_c Y^{ca}|_a \partial^b Y_{ba}|{}^d \right). \quad (2.3.5)$$

In order to analyse the gauge invariances of the action, it is sufficient to use only a decomposition of the various fields under $GL(D)$ and not under $O(1, D-1)$. Thus, we decompose

$$Y^{ab}|_c = X^{ab}|_c + \delta_c^{[a} Z^{b]}, \quad X^{ab}|_a \equiv 0. \quad (2.3.6)$$

The invariance of the Maxwell action under the gauge transformation $\delta A_a = \partial_a \lambda$ is inherited by the new action (2.3.5), whereby the field Z transforms as $\delta Z_a = \partial_a \lambda$, with $X^{ab|}_c$ staying unchanged, *i.e.* the action (2.3.5) can be shown to be invariant under

$$\delta_\lambda Y^{ab|}_c = \delta_c^{[a} \partial^{b]} \lambda. \quad (2.3.7)$$

On the other hand, from the fact that the field $Y^{ab|}_c$ enters the action (2.3.5) only through its divergence $\partial_c Y^{ca|}_b$, the action is manifestly invariant under the following gauge transformation

$$\delta_\Upsilon Y^{ab|}_c = \partial_d \Upsilon^{dab|}_c, \quad \Upsilon^{dab|}_c \equiv \Upsilon^{[dab]|}_c. \quad (2.3.8)$$

Using the invariant antisymmetric symbol of $SL(D)$ to dualise the first two indices of $Y^{ab|}_c$, the decomposition (2.3.6) is tantamount to the following $GL(D)$ -irreducible decomposition

$$\begin{aligned} \tilde{Y}_{a[D-2]|c} &= T_{a[D-2]||c} + \tilde{Z}_{a[D-2]c}, \quad T_{a[D-2]||a} \equiv 0, \quad \tilde{Z}_{a[D-2]c} \equiv \tilde{Z}_{[a[D-2]c]}, \\ \tilde{Y}_{a[D-2]||c} &:= \frac{1}{2} \epsilon_{a[D]} Y^{a[2]|}_c, \quad T_{a[D-2]||c} := \frac{1}{2} \epsilon_{a[D]} X^{a[2]|}_c, \quad \tilde{Z}_{a[D-2]c} := \frac{1}{2} \epsilon_{cba[D-2]} Z^b, \end{aligned} \quad (2.3.9)$$

while the gauge parameter Υ is similarly dualised into

$$\tilde{\Upsilon}_{a[D-3]|c} = \lambda_{a[D-2]||c}^{(1)} + \lambda_{a[D-2]c}^{(2)}, \quad \lambda_{a[D-3]||a}^{(1)} \equiv 0, \quad \lambda_{a[D-3]c}^{(2)} \equiv \lambda_{[a[D-3]c]}^{(2)}. \quad (2.3.10)$$

At this stage, without losing any of the tensorial fields involved, we set $D = 4$ for the sake of clarity and to further explain the gauge structure of the new action in terms of the $GL(4)$ -irreducible dual fields $T_{[2,1]}$ and $Z_{[1]}$. The gauge transformations leaving the action (2.3.5) invariant, with $D = 4$ and keeping the vector field Z_a instead of its Hodge dual $\tilde{Z}_{a[3]}$, now read

$$\begin{aligned} \delta T_{ab||c} &= 2 \partial_{[a} \lambda^{(1)}_{b]c} - 2 \partial_{[a} \lambda^{(2)}_{b]c} + 2 \partial_c \lambda^{(2)}_{ab}, \\ \delta Z_a &= \partial_a \lambda + \partial^b \tilde{\lambda}^{(2)}_{ab}, \quad \tilde{\lambda}^{(2)}_{ab} = \frac{1}{2} \epsilon_{abcd} \lambda^{(2)cd}, \end{aligned} \quad (2.3.11)$$

where $\lambda^{(1)}_{ab} = \lambda^{(1)}_{(ab)}$ and $\lambda^{(2)}_{ab} = \lambda^{(2)}_{[ab]}$.

In terms of the fields $X^{ab|}_c$ and Z_a that we keep for the moment, the equations of motion derived from (2.3.5) are

$$0 = G_{ac|}^b := \frac{1}{2} (\partial^b F_{ac}(Z) + 2 \partial^d \partial_{[c} X_{a]d|}^b), \quad F_{ac}(Z) := 2 \partial_{[a} Z_{c]}. \quad (2.3.12)$$

When the field X is expressed in terms of its dual T , in four dimensions, we have the field equations

$$\partial_b F_{ac}(Z) + \frac{1}{12} [\epsilon_{aduv} \partial_c F^{duv||}_b - \epsilon_{cduv} \partial_a F^{duv||}_b] = 0, \quad F_{abc||d} := 3 \partial_{[a} T_{bc]||d}, \quad (2.3.13)$$

where we note that the curvature $F_{abc||d}$ of T is invariant under the $\lambda^{(1)}$ gauge symmetry. Dualising on the indices ac gives

$$2\partial_d \tilde{F}^{ab}(Z) - \partial_c F^{abc||}_d = 0, \quad \text{where} \quad \tilde{F}^{ab}(Z) := \frac{1}{2} \epsilon^{abcd} F_{cd}(Z). \quad (2.3.14)$$

Antisymmetrising the left-hand side of the equations of motion (2.3.13) in its free indices, one finds

$$\partial_a F^{abc||}_c = 0. \quad (2.3.15)$$

In this equation, only the $GL(4)$ -irreducible field $T_{[2,1]}$ appears and all the symmetries in (2.3.11) are preserved. The above field equation is nothing but the equation

$$G^{(1)}_{a_1 a_2 c||}{}^{a_1 a_2} = 0, \quad (2.3.16)$$

presented in (2.2.8), in the case where $D = 4$. We note that, using the Hodge decomposition whereby a differential p -form can be written as the sum of three terms, one d-exact, one $*d*$ -exact and the last one harmonic:

$$\omega_{[p]} = dp_{[p-1]} + *d*q_{[p+1]} + r_{[p]}, \quad \{ *d*, d \} r_{[p]} = 0,$$

the field Z_a can be set to zero using the λ and $\lambda^{(2)}$ gauge parameters, while its harmonic piece can be obtained by integrating equation (2.3.13), thereby expressing it in terms of the physical components of T . In the gauge where the closed and co-closed parts of Z_a vanish, one cannot use any $\lambda^{(2)}$ gauge parameters anymore and the remaining action and field equations are only invariant under the $\lambda^{(1)}$ gauge symmetry.

3 The three form in eleven dimensions

The eleven dimensional supergravity theory as originally formulated contains the graviton and the three form as its bosonic sector [30]. How to formulate the eleven dimensional action with a six form was discussed in reference [31]. The E_{11} non-linear realisation in eleven dimensions includes the usual fields for the graviton and three form as well as the six form and a field which is the dual of the graviton, but in addition it contains an infinite number of fields with blocks of height nine added, see equation (1.1). Among these fields are the $h_{[9,9,\dots,9,8,1]}$. In this section we will repeat the considerations of the sections two, but for the three form. We will show ! how the alternative dual descriptions of the degrees of freedom usually encoded in the three form arise naturally within the unfolded formulation. We find the equations of motion of the theory when described by any of these dual gauge fields and we will find an infinite set of duality relations that are first order in space-time derivatives and encode the dynamics. As we will discuss in the Conclusions, these relations should be contained in the non-linear realisation based on E_{11} .

3.1 The unfolded representation of the three-form

In what follows we will construct the unfolded representation of the three-form, that is both $SO(1, 10)$ and gauge invariant. We recall that this representation is indecomposable,

but can be mapped via harmonic expansion to Wigner's irreducible unitary representation of $ISO(1, 10)$ for the three form. It can be found following the unfolding procedure given in [19], see also [20,22] and references therein. Unlike in the previous section where we presented the unfolded formulation of Maxwell's theory using standard tensor calculus, in this section we give a more formal and compact account of the unfolded representation using differential form calculus and stress its conceptual basis.

Wigner's unitary irreducible representation of the Poincaré group $ISO(1, 10)$ corresponding to the free, dynamical three-form in eleven dimensions can be mapped to an unfolded module consisting of an infinite set \mathcal{T} of $SO(1, 10)$ -irreducible tensors

$$\mathcal{T} = \{F_{a[4]}, F_{a[4]||b}, F_{a[4]||b(2)}, F_{a[4]||b(3)}, \dots\}, \quad (3.1.1)$$

where the notation $F_{a[4]||b(n)}$ indicates a tensor that is separately antisymmetric in its four indices $\{a_1, a_2, a_3, a_4\}$ and totally symmetric in its n indices $\{b_1, b_2, \dots, b_n\}$. The $SO(1, 10)$ -irreducibility of the tensors $\{F_{a[4]||b(n)}, n = 0, 1, \dots\}$ means that, besides being Young-projected, the tensors are traceless, viz.

$$F_{a[4]||ab(n-1)} \equiv 0, \quad \eta^{a_1 b_1} F_{a[4]||b(n)} \equiv 0, \quad (3.1.2)$$

where we recall our convention that indices at the same position (covariant or contravariant) and with the same Latin label are implicitly symmetrised, or antisymmetrised, according to the context. We note that the difference between an $GL(11)$ and an $SO(1, 10)$ -irreducible tensor is given by the tracelessness property, here the second identity of (3.1.2).

In terms of Young tableau, the tensor $F_{a[4]||b(n)}$ is represented by

a_1	b_1	\dots	b_n
a_2			
a_3			
a_4			

(3.1.3)

The action of the Poincaré group on the infinite set of tensors in (3.1.1) is given by

$$P_b F_{a[4]} = F_{a[4]||b}, \quad P_{b_2} F_{a[4]||b_1} = F_{a[4]||b_1 b_2}, \quad P_{b_3} F_{a[4]||b_1 b_2} = F_{a[4]||b_1 b_2 b_3}, \quad \dots \quad (3.1.4)$$

while the Lorentz generators M_{ab} act diagonally in \mathcal{T} by the usual action. Up to this stage, although we have talked of tensors, we have used no notion of spacetime.

Introducing a spacetime, the action of the translation generators of the Poincaré group on the representation can be explicitly realised by taking them to be differentiation with respect to the space-time coordinates, that is, $P_a = \partial_a$, whereupon equations (3.1.4) take the form

$$\partial_b F_{a[4]} = F_{a[4]||b}, \quad (3.1.5)$$

$$\partial_{b_2} F_{a[4]||b_1} = F_{a[4]||b_1 b_2} , \quad (3.1.6)$$

$$\partial_{b_3} F_{a[4]||b_1 b_2} = F_{a[4]||b_1 b_2 b_3} , \quad (3.1.7)$$

\vdots

The infinite set of differential equations (3.1.5-7) can be compactly written upon introducing Grassmann odd (resp. even) vector oscillators θ^a (resp. u^a) and forming the master field

$$F(x; \theta, u) = \sum_{n=0}^{\infty} \frac{1}{4!n!} F_{a[4]||b(n)}(x) \theta^{a_1} \dots \theta^{a_4} u^{b_1} \dots u^{b_n} . \quad (3.1.8)$$

It is also advantageous to write everything in terms of differential forms, by using the total exterior derivative $d = dx^\mu \partial_\mu$, taking the $F_{a[4]||b(n)}$ to be zero forms and introducing the one-form

$$h^a := dx^\mu \delta_\mu^a \quad (3.1.9)$$

for Minkowski spacetime in Cartesian coordinates. In this setting, the infinite set of differential equations (3.1.5-7) given above can be written in the form

$$[d - i h^a \rho_\tau(P_a)] F(x; \theta, u) = 0 , \quad (3.1.10)$$

where the translation generators are now represented on the master field as follows:

$$\rho_\tau(P_a) = (-i) \frac{\partial}{\partial u^a} . \quad (3.1.11)$$

Explicitly, equations (3.3.5)–(3.1.7) now read

$$dF_{a[4]} = h^c F_{a[4]||c} , \quad (3.1.12)$$

$$dF_{a[4]||b} = h^c F_{a[4]||bc} , \quad (3.1.13)$$

$$dF_{a[4]||b(2)} = h^c F_{a[4]||b(2)c} . \quad (3.1.14)$$

Taking into account the $GL(11)$ irreducibility conditions, given in equation (3.1.2), of the tensor on the right-hand side of equation (3.1.5), one derives the relation

$$\partial_a F_{a[4]} = 0 , \quad (3.1.15)$$

which is locally solved, as usual, by $F_{[4]} = dA_{[3]}$, introducing a three-form potential and its four-form field strength

$$A_{[3]} = \frac{1}{6} h^{a_1} \wedge h^{a_2} \wedge h^{a_3} A_{a[3]} , \quad F_{a[4]} := \frac{1}{24} h^{a_1} \wedge \dots \wedge h^{a_4} F_{a[4]} . \quad (3.1.16)$$

We are using the notation that a number in square brackets without being accompanied by a letter denotes the degree of the form that the field belongs to, that is, $A_{[3]}$ is a form

of degree three. The zero-form tensor $F_{a[4]}$ are thus the components of the four-form field strength $F_{[4]} = dA_{[3]}$. As usual the gauge field $A_{[3]}$ is defined up to the exterior derivative of a two-form potential, namely

$$A_{[3]} \sim A_{[3]} + d\Lambda_{[2]} . \quad (3.1.17)$$

On the other hand, recalling that $F_{a[4]||b}$ is not only $GL(11)$ but also $SO(1, 10)$ irreducible, as given in equation (3.1.2), one derives the equation

$$\partial^a F_{a[4]} = 0 , \quad (3.1.18)$$

which together with equation (3.1.16), is the field equation of a dynamical three-form. We also note that the other equations (3.1.13), (3.1.14) *etc.* can be solved one after the others. They express the tensors $F_{a[4]||b(n)}$ as the higher gradients of the tensors $F_{a[4]||b(m)}$ for $m < n$ and so in terms of the *on-shell* dynamical three-form $A_{[3]}$:

$$F_{a[4]||b(n)} = 4\partial_{b_1}\partial_{b_2}\dots\partial_{b_n}\partial_{[a_1}A_{a_2a_3a_4]} . \quad (3.1.19)$$

We note that the $SO(1, 10)$ properties of $F_{a[4]||b(n)}$ are ensured by the equations of motion of the three form and the fact that partial derivatives commute.

To summarise, the irreducible unitary representation of equation (3.1.1) contains components that are individually subject to $SO(1, D-1)$ irreducibility conditions and once we take the space-time translations to be realised by space-time differentiation these conditions imply the well known equation of motion for a three form. This is a purely algebraic way of encoding the field equations and Bianchi identities of a dynamical three-form, a characteristic of unfolded dynamics.

The underlying algebraic structure, captured by (3.1.12)–(3.1.14) together with $dA_{[3]} = \frac{1}{24}h^{a_1}\wedge\dots\wedge h^{a_4}F_{a[4]}$ and $dh^a = 0$, is known as a free differential algebra and makes sense on a base manifold of arbitrary dimension. Its initial data is given by the gauge functions for $A_{[3]}$ and the vielbeins h^a together with the infinite set of constants provided by the zero-forms at a given point p_0 of the manifold. In particular, in eleven dimensions, the infinite set of zero-forms in \mathcal{T} at a point p_0 with Cartesian coordinates x_0^μ , together with the differential equations (3.1.10), give the necessary data that enables one to reconstruct an on-shell, dynamical three-form around that point p_0 using the Taylor expansion

$$A_{a[3]}(x) = A_{a[3]}(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} (x - x_0)^{b_1} \dots (x - x_0)^{b_n} F_{a[3]b||b(n-1)}(x_0) . \quad (3.1.20)$$

We would like to make some comments on gauge fixing. In the light-cone coordinates $x^\mu = (x^-, x^+, x^i)$ we can choose the Lorentz frame in which the momentum is $k_\mu = (k_-, k_+ = 0, k_i = 0)$. Then at the point p_0 , the components A_{-jk} and A_{-+j} can be set to zero by fixing the gauge in equation (3.1.17) using the gauge parameters λ_{ij} and λ_{+i} . Furthermore, the components A_{+ij} are gauge-invariant and zero on-shell as the field

equation is given by $k_- A_{+ij} = 0$. As a result one finds that the three-form potential has all its components vanishing except for the purely transverse ones, for which

$$A_{ijk}(x_0) = \frac{1}{k_-} F_{-ijk}(x_0) . \quad (3.1.21)$$

Consequently, all the derivatives of the three-form, when evaluated in momentum space and in the chosen Lorentz frame, are therefore given by all the powers of k_- times the Fourier transform of $A_{ijk}(x)$ and they transform in the following representation

—	—	...	—
i			
j			
k			

(3.1.22)

These coincide with all the non-vanishing on-shell derivatives of the field strength. This discussion follows the general arguments given in references [18] (for related discussions, see [27]) and it is the equivalent, for the three-form, of the Petrov decomposition of a metric in Riemannian geometry.

We next note how the gauge-for-gauge transformations, $\delta\lambda_{[2]} = dC_{[1]}$, act. In our chosen Lorentz frame the only gauge transformations that have a non-trivial gauge-for-gauge transformation are λ_{-i} and λ_{-+} . These are also the only gauge parameters which we did not use so far. They are subject to transformations that involve the components C_i and C_+ and these can be used to set these gauge parameters to zero, that is set $\lambda_{-i} = 0 = \lambda_{-+}$. We note that the component C_- can be set to zero by the gauge-for-gauge-for-gauge parameter.

Another, alternative and Lorentz-covariant way of analysing the physical content of the equations consists in Taylor expanding the gauge (and higher reducibility) parameters, the three-form components as well as the field strength, all evaluated on-shell, and comparing all the coefficients of the various powers of $(x - x_0)$ at the point p_0 . One sees that the constants $A_{abc}(x_0)$ can be set to zero by the constants $\partial_{[a}\lambda_{bc]}(x_0)$ (the latter not being constrained by the reducibility transformations). Similarly, at first order in the derivatives of the three-form, the constants $\partial_{(a}A_{b)cd}(x_0)$ can be set to zero by the constants $\partial_a\partial_{[b}\lambda_{cd]} + \partial_b\partial_{[a}\lambda_{cd]}$ whereas the constants $\partial_{[a}A_{bcd]}(x_0)$ are identified (up to a constant, irrelevant factor) with the constants $F_{abcd}(x_0)$, *etc.* The outcome of this procedure is that all the derivatives $\partial_{c_n} \dots \partial_{c_1} A_{a[3]}(x_0)$ of the three-form at the point p_0 are set equal to the on-shell derivatives $\partial_{(c_n} \dots \partial_{c_2} F_{c_1)a[3]}(x_0)$, thereby explaining (3.1.20). This way of counting physical degrees of freedom on-shell is the one adopted in unfolded dynamics [19].

It is well known that rather than describe the degrees of freedom by a three form one can use a 6-form potential and we now explain this from the unfolded viewpoint. We begin with the relation

$$F^{a[7]} := \frac{1}{4!} \epsilon^{a[7]b[4]} F_{b[4]} , \quad (3.1.23)$$

and transfer the properties of the unfolded dynamics of the three form given in equations (3.1.5-7) to corresponding equations for the six form. The first unfolded equation (3.1.5) transforms in the $[4, 1]$ -irrep of $SO(1, 10)$ and the resulting divergenceless property of $F_{a[4]||b}$ implies that $F^{a[7]}$ is d-closed:

$$0 = \partial^a F_{ac[3]} \quad \Leftrightarrow \quad \partial^a F^{a[7]} = 0 , \quad (3.1.24)$$

while the $GL(11)$ -irreducibility of $F_{a[4]||b}$, that is the Bianchi identity of $F_{a[4]}$, implies that $F^{a[7]}$ is divergenceless:

$$\partial_a F_{a[4]} \equiv 0, \quad \Leftrightarrow \quad \partial_b F^{ba[6]} = 0 . \quad (3.1.25)$$

By the usual Poincaré lemma, equation (3.1.24) implies that $F^{a[7]}$ can locally be written as

$$F^{a[7]} = 7\partial^a A^{a[6]} . \quad (3.1.26)$$

Thus we find the usual exchange the equations of motion with the Bianchi identities in equations (3.1.24) and (3.1.25).

We now define $F_{a[7]||b}$ by

$$F_{a[7]||b} := \partial_b F_{a[7]} . \quad (3.1.27)$$

By virtue of equations (3.1.24) and (3.1.25), $F_{a[7]||b}$ is an irreducible $SO(1, 10)$ tensor as it is $GL(11)$ -irreducible ($F_{a[7]||a} = 0$) and traceless ($F_{a[6]b||}{}^b = 0$). Completing the unfolding of the dual linearised 6-form yields the following tower of tensors

$$\tilde{\mathcal{T}} = \{F_{a[7]||b(n)} , \quad n = 0, 1, \dots, \} . \quad (3.1.28)$$

The action of the Poincaré generators P_c on the tensors in $\tilde{\mathcal{T}}$ is given by

$$P_c F_{a[7]||b(n)} = F_{a[7]||cb(n)} = \partial_c F_{a[7]||b(n)} . \quad (3.1.29)$$

It follows from (3.1.23) and the above conventions for the action of the Poincaré translations that the tensors in $\tilde{\mathcal{T}}$ of equation (3.1.28) and those in \mathcal{T} of equation (3.1.1) are related by

$$F_{a[7]||b(n)} = \frac{1}{4!} \epsilon^{a[7]c[4]} F_{c[4]||b(n)}, \quad n = 1, 2, \dots \quad (3.1.30)$$

The tensors in $\tilde{\mathcal{T}}$ are traceless as result of the relation

$$F^{a[6]c||}{}_{cb(n-1)} = \frac{1}{4!} \epsilon^{a[6]cd[4]} F_{d[4]||cb(n-1)} = 0 , \quad (3.1.31)$$

and are $GL(11)$ -irreducible as a consequence of

$$\epsilon^{a[8]d[3]} F_{a[7]||ab(n-1)} = \frac{1}{4!} \epsilon^{a[8]d[3]} \epsilon_{a[7]c[4]} F^{c[4]||}{}_{ab(b-1)} = 7! F^{d[3]a||}{}_{ab(n-1)} = 0 . \quad (3.1.32)$$

Hence the tensors $\tilde{\mathcal{T}}$ of equation (3.1.28) belong to the $SO(1, 10)$ Young tableau

a_1	b_1	\dots	b_n
\vdots			
a_6			
a_7			

(3.1.33)

We can collect the tensors $\tilde{\mathcal{T}}$ into a single object

$$F(x; \theta, u) = \sum_{n=0}^{\infty} \frac{1}{7!n!} F_{a[7]||b(n)}(x) \theta^{a_1} \dots \theta^{a_7} u^{b_1} \dots u^{b_n}, \quad (3.1.34)$$

for which equation (3.1.29) takes the form

$$[d - i h^a \rho_{\tilde{\tau}}(P_a)] F(x; \theta, u) = 0, \quad \text{where} \quad \rho_{\tilde{\tau}}(P_a) = (-i) \frac{\partial}{\partial u^a} = \rho_{\tau}(P_a). \quad (3.1.35)$$

Although action principles are usually part of the definition of an unfolded system, it is nevertheless instructive to consider a parent action from which one can find both the action for the three gauge form and that for the six form gauge field:

$$S[A_{[3]}, F_{[7]}] = \int (dA_{[3]} \wedge F_{[7]} - \frac{1}{8} F_{[7]} \wedge *F_{[7]}), \quad (3.1.36)$$

where $F_{[7]}$ and $A_{[3]}$ are independent fields. Extremising it with respect to $A_{[3]}$ gives $dF_{[7]} = 0$ and so $F_{[7]} = dA_{[6]}$; substituting this back in $S[A_{[3]}, F_{[7]}]$, gives the standard action $S[A_{[6]}] \propto \int dA_{[6]} \wedge *dA_{[6]}$. The equation of motion for F_7 gives $F_7 \propto *dA_3$ and substituting back we find the standard action for the three form.

Alternatively, one can start from the Palatini formulation for the 3-form,

$$S[A_{[3]}, F^{a[4]}] = \int \frac{1}{7!} \epsilon^{b[4]c[7]} h_{c_1} \wedge \dots \wedge h_{c_7} (dA_{[3]} + \frac{1}{8} h_{c_1} h_{c_2} h_{c_3} h_{c_4} F^{c[4]}) F_{b[4]}, \quad (3.1.37)$$

where $F^{a[4]}$ is a zero-form and is an independent field and we recall that h_c is defined in equation (3.1.9). Defining

$$F_{[7]} := \frac{1}{7!} \epsilon^{b[4]c[7]} F_{b[4]} h_{c_1} \wedge \dots \wedge h_{c_7}, \quad (3.1.38)$$

the Palatini action (3.1.37) becomes identical to the action (3.1.36). The latter action will be used in section 3.3 where we shall generalise the action principle given above for Maxwell theory to the case of the three form in eleven dimension and in the frame-like formulation.

3.2 Further dualisation of the three form

It is well-known that rather than express the dynamics of the bosonic non-gravitational degrees of freedom of eleven dimensional supergravity by a three-form gauge field one can instead use a six-form gauge field $A_{[6]}$, whose curvature $F_{[7]}$, at the linearised level, is just the Hodge dual of $F_{[4]}$. As explained in the introduction, the non-linear realisation of the Kac-Moody algebra E_{11} leads not only to the usual fields of eleven dimensional supergravity as well as a six form and dual graviton field, but also to the infinite set of fields of equation (1.1) which were proposed to be equivalent ways of describing the dynamics [6]. In this section we will show how the next field on the duality chain of equation (1.1), the gauge field $A_{[9,3]}$, arises and we give its linearised dynamics. The duality relation involving the fields in the gravity sector was sketched in reference [14] and some indications that one might be able to do this for any massless particle were discussed in [24].

As we explained for Maxwell theory in the previous section, one can dualise any of the curvature tensors that occur in the unfolded formulation. Hence, instead of dualising the first tensor in the set \mathcal{T} in (3.1.1), one may dualise the second tensor $F_{[4,1]}$ on its second column:

$$G^{b[10]}{}_{||a[4]} = \epsilon^{b[10]c} F_{a[4]||c} , \quad (3.2.1)$$

or equivalently

$$F_{a[4]||b} = -\frac{1}{10!} \epsilon_{bc[10]} G^{c[10]}{}_{||a[4]} . \quad (3.2.2)$$

We can now find what the constraints on $F_{a[4]||b}$ imply for $G_{b[10]||a[4]}$. Taking the trace of (3.2.2) and using the second equation in (3.1.2) we find that indeed,

$$G_{b[10]||ba[3]} = 0 , \quad (3.2.3)$$

while using the first equation in (3.1.2) and acting with $\epsilon^{a[4]bd[6]}$ on equation (3.2.2) we find the quartic trace constraint

$$G^{b[6]a[4]}{}_{||a[4]} \equiv (\text{Tr}_{12})^4 G_{[10,4]} = 0 . \quad (3.2.4)$$

The presence of a higher order trace condition is unusual when compared to the standard formulation of particle dynamics including Fronsdal's higher-spin dynamics.

Equation (3.2.3) implies that the tensor $G_{a[10]||b[4]}$ is an irreducible $GL(11)$ tensor of type $[10|4]$, but it is not an $SO(1, D-1)$ -irreducible tensor as it does not satisfy a single trace condition. We note that, as usual, the Bianchi identities and field equations get swapped under the dualisation.

We would now like to look at the differential constraints on $G_{b[10]||a[4]}$ that arise from the differential constraints on $F_{a[4]||c}$ of equations (3.1.5-7). The first of these equations implies that $\partial_a F_{a[4]||b} = 0$ which using equation (3.2.2) in turn implies that

$$\partial_a G^{b[10]}{}_{||a[4]} = 0 . \quad (3.2.5)$$

As we did for the Maxwell case we can continue taking more space-time derivatives of the field strength $G_{b[10]||a[4]}$ to find an infinite set of tensors $\{G_{a[10]||b[4]||b(n)}, n = 0, 1, \dots\}$.

Using similar arguments we can transfer the properties of $F_{a[4]||b(n)}$ to those new tensors to find that $\{G_{a[10]||b[4]||b(n)}, n = 0, 1, \dots\}$ are $GL(11)$ -irreducible and obey the trace constraints

$$(\text{Tr}_{12})^4 G_{[10,4,1,\dots,1]} = 0, \quad \text{Tr}_{1i} G_{[10,4,1,\dots,1]} = 0 = \text{Tr}_{2i} G_{[10,4,1,\dots,1]}, \quad i \in \{3, \dots, n\}. \quad (3.2.6)$$

The notation $(\text{Tr}_{ij})^n$ used here means that one takes n traces on the columns i and j .

Equation (3.2.5), combined with the $GL(11)$ irreducibility of $G^{b[10]}_{||a[4]}$ implies, using the generalised Poincaré lemma [28], that it can locally be written as

$$G^{b[10]}_{||a[4]} = \partial_a \partial^b A^{b[9]}_{||a[3]}, \quad (3.2.7)$$

where the $GL(11)$ -irreducible tensor gauge field $A_{b[9]||a[3]}$ is defined up to the gauge transformation

$$\delta A_{a[9]||b[3]} = 9 \partial_a \Lambda^{(1)}_{a[8]||b[3]} + 3 (\partial_b \Lambda^{(2)}_{a[9]||b[2]} + \frac{9}{7} \partial_a \Lambda^{(2)}_{a[8]b||b[2]}) , \quad (3.2.8)$$

with the two gauge parameters being $GL(11)$ -irreducible with type $\Lambda^{(1)}_{[8,3]}$ and $\Lambda^{(2)}_{[9,2]}$. We note that there are no algebraic trace constraints on $A_{[9,3]}$, nor on its gauge parameters.

Remembering the expression $F_{a[4]||b} = \partial_b \partial_a A_{a[3]}$, the definition (3.2.1) of $G_{a[10]||b[4]}$ and the relation (3.2.7) give us the following duality relation:

$$\partial^a A^{a[9]||}_{b[3]} = \epsilon^{a[10]c} \partial_c A_{b[3]} + \partial_b \Xi^{a[10]}_{||b[2]}. \quad (3.2.9)$$

which is the analog of (2.2.38). We first note that $\Xi^{a[10]}_{||b[2]}$ decomposes into

$$\Xi^{a[10]}_{||b[2]} = \Xi^{(1)a[10]}_{||b[2]} + \epsilon^{a[10]}_b \Xi^{(2)}_b \quad (3.2.10)$$

and that a gauge transformation $A_{b[3]} \rightarrow A_{b[3]} + \partial_b \lambda_{b[2]}$ with $\lambda_{b[2]} = -x_b \Xi^{(2)}_b$ enables one to eliminate the $\Xi^{(2)}$ component of Λ . Having done that, the equation (3.2.9) is now understood with a field $\Xi_{a[10]||b[2]}$ obeying $\Xi_{a[10]||ab} = 0$. We can now reformulate this equation in the same manner as we did for equation (2.2.38). By shifting the arbitrary field $\Xi^{a[10]||}_{b[2]}$ in an appropriate way, we can recast the equation in the form

$$\partial^a A^{a[9]||}_{b[3]} = \epsilon^{a[10]b} 4 \partial_b A_{b[3]} + \partial_b \Xi^{a[10]||}_{b[2]}. \quad (3.2.11)$$

Multiplying by $\epsilon^{a[10]e}$ and tracing on b_1 and e , we find that $\partial^a \Xi^{a[10]||}_{b[2]} = 0$ implying that $\Xi^{a[10]||}_{b[2]} = \partial^a \Xi^{a[9]||}_{b[2]}$. Using this result equation (3.2.9) now becomes

$$\partial^a A^{a[9]||}_{b[3]} = \epsilon^{a[10]c} 4 \partial_{[c} A_{b[3]}] + \partial_b \partial^a \Xi^{a[9]||}_{b[2]}. \quad (3.2.12)$$

We recognise the last term as a gauge transformation of the field $A^{a[9]||}_{b[3]}$. Alternatively, the above equation can be made fully gauge invariant by giving a shift symmetry to the field Ξ under $\Lambda^{(2)}_{[9,2]}$.

We now give an action principle for the $A_{b[9]||a[3]}$ potential that correctly describes the degrees of freedom of a massless three-form. The procedure was proposed in [14], which itself was inspired from [6,9]. We start with the three-form and follow the analog of the procedure for the Maxwell field spelled out in Section 2.3. To this end, we take the usual action $S[A_{[3]}]$ for a three form and integrate by parts, ignoring boundary terms:

$$-\frac{1}{4!} \int d^D x \partial_a A_{a[3]} \partial^a A^{a[3]} = -\frac{1}{3!} \int d^D x \left(\partial_b A_{a[3]} \partial^b A^{a[3]} + 3 \partial^b A_{ba[2]} \partial_c A^{ca[2]} \right) . \quad (3.2.13)$$

We then introduce the following parent action, that features two independent fields, $P_{b|a[3]}$ and $Y^{b[2]||a[3]}$:

$$S[P, Y] = -\frac{1}{3!} \int d^D x \left(P_{b|a[3]} \partial_c Y^{cb|a[3]} + P_{b|a[3]} P^{b|a[3]} + 3 P^{b|}_{ba[2]} P_{c|}{}^{ca[2]} \right) . \quad (3.2.14)$$

Varying the action $S[P, Y]$ with respect to the field $Y^{a[2]||b[3]}$ gives the equation $\partial_{b_1} P_{b_2|a[3]} = 0$ which implies that $P_{b|a[3]} = \partial_b A_{a[3]}$. Substituted inside the action, we reproduce the action (3.2.13). On the other hand, as the field $P_{b|a[3]}$ is auxiliary one can express it in terms of Y via its equation of motion, namely

$$2P^{b|a[3]} = -\partial_c Y^{cb|a[3]} - \frac{3}{D-1} \eta^{ba} \partial_c Y^{cd|}{}_d{}^{a[2]} , \quad (3.2.15)$$

and substitute for it into the parent action, thereby yielding a daughter action $S[Y^{b[2]||a[3]}]$ expressed solely in terms of the field Y :

$$S[Y^{cb|a[3]}] = \frac{1}{4!} \int d^D x \left(P_{b|a[3]} \partial_c Y^{cb|a[3]} + \partial_c Y^{cb|a[3]} \partial^e Y_{eb|a[3]} \right. \\ \left. - \frac{3(5D^2 - 11D + 7)}{(D-1)^2} \partial_c Y^{cb|}_{ba[2]} \partial^e Y_{ed|}{}^{da[2]} \right) . \quad (3.2.16)$$

Setting $D = 11$, one can then dualise $Y^{a[2]||b[3]}$ on its first two indices, and decompose

$$\tilde{Y}_{a[9]||b[3]} = \frac{1}{2} \epsilon_{a[9]c[2]} Y^{c[2]||b[3]} = A_{a[9]||b[3]} + B_{a[9]b||b[2]} + \epsilon_{a[9]b[2]} C_b , \quad (3.2.17)$$

so as to produce the $GL(11)$ -irreducible field $A_{a[9]||b[3]}$ satisfying $A_{a[9]||ab[2]} \equiv 0$, as well as $B_{a[10]||b[2]}$ (satisfying $B_{a[10]||ab} \equiv 0$) and C_a which are analogs of the field \tilde{Z} in Equation (2.3.9). Because the field $Y^{cb|a[3]}$ enters the action only through its divergence $\partial_c Y^{cb|a[3]}$, the action is invariant under the following gauge transformations

$$\delta Y^{b[2]||a[3]} = \partial_c \Upsilon^{c[3]||a[3]} , \quad (3.2.18)$$

where the gauge parameter Υ is antisymmetric in its two groups of indices. Upon dualising the parameter Υ , one gets the following $GL(11)$ -irreducible gauge parameters

$$\frac{1}{3!} \epsilon_{c[3]} \Upsilon^{c[3]||a[3]} \longrightarrow \{ \Lambda_{a[8]||b[3]}^{(1)} , \Lambda_{a[9]||b[2]}^{(2)} , \Lambda_{a[10]||b}^{(3)} , \Lambda^{(4)} \} . \quad (3.2.19)$$

The field $A_{a[9]||b[3]}$ will then transform as in (3.2.8), while the gauge transformation of the field $B_{a[10]||b[2]}$ will involve the gradient of the parameters $\Lambda^{(2)}$ and $\Lambda^{(3)}$. Finally, the vector field C_a will transform with the gradient of $\Lambda^{(4)}$.

We note that the action also possesses the gauge symmetry involving the two-form gauge parameter $\lambda_{a[2]}$ inherited from the original three-form $A_{a[3]}$. This will be discussed in the next section 3.3, where we use the frame-like formalism that brings in a better insight into the gauge structure. On-shell, the gauge field $A_{a[9]||b[3]}$ will obey the equation (3.2.4) discussed above.

3.3 Unfolded description containing the $A_{[9,3]}$ form

In this section we wish to construct the unfolded formulation of the dynamics for the $A_{[9,3]}$ form, that is a set of first order differential equations that contain the gauge field $A_{a[9]||b[3]}$ and that assumes the form of a free differential algebra. This will contain the manifestly Lorentz covariant and gauge-invariant infinite-dimensional representation of $ISO(1, D-1)$ constructed from the field strength, $G_{a[10]||b[4]}$, and all of its higher on-shell derivatives. It also contains the gauge field $A_{a[9]||b[3]}$ through an appropriate frame-like, or Cartan-like, connection. In the next subsection 3.4, we will build an action principle for the $A_{a[9]||b[3]}$ potential, but this time facilitated by the use of the frame-like description that we first derive on-shell in the present subsection.

We first introduce, following [20], the connection-like objects

$$\{e_{[9]}^{a[3]}, \omega_{[3]}^{a[10]}\}. \quad (3.3.1)$$

The indices in square brackets without a label, i.e. $[3]$ and $[9]$, denote the form degree of the objects, for example $e_{[9]}^{a[3]}$ is a nine form that carries three antisymmetrised tangent indices and so can be written in more usual notation as $\frac{1}{9!} h^{b_1} \wedge \dots \wedge h^{b_9} e_{b_1 \dots b_9}^{a_1 a_2 a_3}$. The field $\omega_{[3]}^{a[10]}$ is a three form that carries ten antisymmetrised tangent indices. It is important to note that the objects of equation (3.3.1) are not subject to any $GL(D)$ irreducibility conditions. By analogy with the vielbein formulation of general relativity, we may think of $e_{[9]}^{a[3]}$ as a generalised vielbein and $\omega_{[3]}^{a[10]}$ as a generalised spin-connection. As the field $e_{[9]}^{a[3]}$ is not $GL(D)$ irreducible, only one of its irreducible components can be identified with the gauge potential $A_{b[9]||a[3]}$ that we considered in section 3.2; the precise identification will be discussed below.

The differential forms of equation (3.3.1) are required to satisfy the differential equations

$$de_{[9]}^{a[3]} + h^{b_1} \wedge \dots \wedge h^{b_7} \wedge \omega_{[3]}^{a[3]}{}_{b[7]} = 0, \quad (3.3.2)$$

$$d\omega_{[3]}^{a[10]} + h^{c_1} \wedge \dots \wedge h^{c_4} G^{a[10]}{}_{c[4]} = 0, \quad (3.3.3)$$

where by assumption $G^{a[10]}{}_{c[4]}$ is the zero-form that appeared in (3.2.1). It obeys the $GL(D)$ irreducibility conditions and the higher-trace constraints of equations (3.2.3), (3.2.4). As discussed (3.2.5), this zero-form is the first member of an infinite set of zero-forms obeying the following first-order differential constraints:

$$dG^{a[10]||b[4]} + h_c G^{a[10]||b[4]||c} = 0, \quad (3.3.4)$$

$$dG^{a[10]||b[4]||c} + h_c G^{a[10]||b[4]||c(2)} = 0 , \quad (3.3.5)$$

\vdots

$$dG^{a[10]||b[4]||c(n)} + h_c G^{a[10]||b[4]||c(n+1)} = 0 , \quad n = 2, 3, \dots \quad (3.3.6)$$

The equations (3.3.2)–(3.3.6) together with $dh^a = 0$ form a free differential algebra and provides the unfolded description of the dual $A_{[9,3]}$ metric-like gauge field.

The gauge transformations of the system (3.3.2)–(3.3.3) are

$$\delta_\epsilon e_{[9]}^{a[3]} = d\epsilon_{[8]}^{a[3]} + h_{a_1} \wedge \dots \wedge h_{a_7} \wedge \epsilon_{[2]}^{a[10]} = 0 , \quad (3.3.7)$$

$$\delta_\epsilon \omega_{[3]}^{a[10]} = d\epsilon_{[2]}^{a[10]} . \quad (3.3.8)$$

The algebraic, Stückelberg-like, gauge transformations on $e_{[9]}^{a[3]}$, that is those contained in $\epsilon_{[2]}^{a[10]}$, can be used to gauge away certain components of $e_{[9]}^{a[3]}$. The $GL(11)$ -irreducible decompositions of $e_{[9]}^{a[3]}$ and $\epsilon_{[2]}^{a[10]}$ are respectively given by

$$[9] \otimes [3] \cong [11, 1] \oplus [10, 2] \oplus [9, 3] , \quad \text{and} \quad (3.3.9)$$

$$[10] \otimes [2] \cong [11, 1] \oplus [10, 2] . \quad (3.3.10)$$

Therefore, after using all the algebraic gauge symmetries, the remaining components in $e_{[9]}^{a[3]}$ are contained in the $GL(11)$ -irreducible gauge field $A_{a[9]||b[3]}$. Thus we make the connection with the equations of motion of section 3.2 which involved the $GL(D)$ -irreducible gauge field $A_{a[9]||b[3]}$. The connection $\omega_{[3]}^{a[10]}$ possesses two $GL(11)$ -irreducible pieces: $[10, 3] \oplus [11, 2]$. However, it is determined from the “zero-torsion” equation (3.3.2) by the first derivatives of the components of $\epsilon_{[9]}^{a[3]}$ that can be reduced (or gauge-fixed) to its $A_{a[9]||b[3]}$ part. As a result we find that only the $[10, 3]$ irreducible component of $\omega_{[3]}^{a[10]}$ remains that we denote by $\tilde{\omega}_{a[10]||b[3]}$.

In summary, so far, Equations (3.3.4)–(3.3.6) constrain a tower of manifestly Lorentz-covariant and gauge-invariant zero forms $\{G^{(n)} , n = 0, 1, \dots\}$ such that these can be expanded in terms of the unitary and irreducible massless representation of $ISO(1, D-1)$ that describes the degrees of freedom propagated by the original three form gauge field. This is simply a consequence of the fact that the field strength $G^{a[10]||b[4]}$ is by assumption expressed in terms of the field strength $F_{a[4]}$ via (3.2.1), so the representation appearing in (3.3.4)–(3.3.6) is equivalent to the representation built on the field strength $F_{a[4]}$ contained in equation (3.1.1). Equations (3.3.2) and (3.3.3) glue the zero-form tower to the gauge field $e_{[9]}^{a[3]}$ thanks to the introduction of the generalised spin connection $\omega_{[3]}^{a[10]}$ so as to write the full system as a free differential algebra. We will show later in this section how to reproduce an equivalent dynamics from an action principle involving the fields in (3.3.1) with some additional zero-forms.

In order to make contact with the gauge parameters of the metric-like $A_{[9,3]}$ gauge fields, we note that, as is typical for p -form systems such as a nine-form and a 3-form, the gauge transformations admit reducibility transformations. The complete family of gauge-for-gauge p -form parameters, which are not $GL(D)$ irreducible, is given by:

$$\{\epsilon_{[9-i]}^{a[3]}\} , \quad i = 1, 2, \dots, 9 \quad (3.3.11)$$

and

$$\{\epsilon_{[3-j]}^{a[10]}\}, \quad j = 1, 2, 3 \quad (3.3.12)$$

with transformation rules

$$\delta\epsilon_{[8]}^{a[3]} = d\epsilon_{[7]}^{a[3]} + h_{a_1} \wedge \dots \wedge h_{a_7} \wedge \epsilon_{[1]}^{a[10]}, \quad \delta\epsilon_{[7]}^{a[3]} = d\epsilon_{[6]}^{a[3]} + h_{a_1} \wedge \dots \wedge h_{a_6} \wedge \epsilon_{[0]}^{a[10]}, \quad (3.3.13)$$

$$\delta\epsilon_{[2]}^{a[10]} = d\epsilon_{[1]}^{a[10]}, \quad \delta\epsilon_{[1]}^{a[10]} = d\epsilon_{[0]}^{a[10]}, \quad \delta\epsilon_{[0]}^{a[10]} = 0, \quad (3.3.14)$$

and

$$\delta\epsilon_{[6]}^{a[3]} = d\epsilon_{[5]}^{a[3]}, \quad \delta\epsilon_{[5]}^{a[3]} = d\epsilon_{[4]}^{a[3]}, \dots, \quad \delta\epsilon_{[1]}^{a[3]} = d\epsilon_{[0]}^{a[3]}, \quad \delta\epsilon_{[0]}^{a[3]} = 0. \quad (3.3.15)$$

The gauge-for-gauge parameter $\epsilon_{[1]}^{a[10]}$ can be used to gauge away parts of the parameter $\epsilon_{[8]}^{a[3]}$. Both are $GL(11)$ reducible and can be decomposed into the $GL(11)$ representations as follows

$$[10] \otimes [1] \cong [11] \oplus [10, 1], \quad [8] \otimes [3] \cong [11] \oplus [10, 1] \oplus [9, 2] \oplus [8, 3] \quad (3.3.16)$$

As this decomposition makes clear we can gauge away two components leaving the gauge parameter $\epsilon_{[8]}^{a[3]}$ to contain only the $GL(11)$ -irreducible representation $[9, 2] \oplus [8, 3]$. Making the appropriate $GL(11)$ projection on equation (3.3.7), we find that the gauge transformation of the $A_{[9,3]}$ potential takes the form :

$$\delta_\epsilon A_{a[9]||}{}^{b[3]} = 9 \partial_a \epsilon_{a[8]||}{}^{b[3]} + 3 (\partial^b \epsilon_{a[9]||}{}^{b[2]} + \frac{9}{7} \partial^b \epsilon_{a[8]||}{}^{b||b}{}_a), \quad (3.3.17)$$

thereby making contact with (3.2.8).

When equations (3.3.2)-(3.3.3) are reduced to the remaining $GL(11)$ -irreducible components $A_{a[9]||}{}^{b[3]}$ and $\tilde{\omega}_{a[10]||}{}^{b[3]}$ of $e_{[9]}^{a[3]}$ and $\omega_{[3]}^{a[10]}$, they become

$$\tilde{\omega}_{a[10]||}{}^{b[3]} = \partial_a A_{a[9]||}{}^{b[3]}, \quad (3.3.18)$$

$$\partial^b \tilde{\omega}_{a[10]||}{}^{b[3]} = G_{a[10]||}{}^{b[4]}. \quad (3.3.19)$$

The expression of the field strength in terms of the gauge field is given by

$$\partial^b \partial_a A_{a[9]||}{}^{b[3]} = G_{a[10]||}{}^{b[4]}, \quad (3.3.20)$$

which agrees with equation (3.2.7). It is easy to see that it is invariant under the gauge transformations (3.3.17).

3.4 First-order frame-like action for the $A_{[9,3]}$ field

We now follow the general procedure explained in [24], whose discussion for the spin-2 case was already given in [29]. The action, just like the one given at the end of section 3.3,

is a parent action in the sense that it contains both the three form and the $A_{[9,3]}$ gauge field. The difference between this action and the one presented in section 3.3 is that we will now use the frame-like vantage point developed above for the gauge field $A_{a[9]||b[3]}$. We start from the action principle for the three-form, written in the Palatini formulation presented at the end of section 3.1 and that we repeat here for convenience:

$$S[A_{[3]}, F^{a[4]}] = \int_{\mathcal{M}_{11}} \frac{1}{7!} \epsilon^{b[4]c[7]} h_{c_1} \wedge \dots \wedge h_{c_7} (dA_{[3]} + \frac{1}{8} h_{c_1} h_{c_2} h_{c_3} h_{c_4} F^{c[4]}) F_{b[4]}, \quad (3.4.1)$$

where $A_{[3]}$ and $F_{a[4]}$ are independent fields. We next introduce the parent action

$$S^P[A_{[3]}, F_{a[4]}, t_{[1]}^{a[3]}, e_{[9]}^{a[3]}] = \int_{\mathcal{M}_{11}} \left[\frac{1}{7!} \epsilon^{b[4]c[7]} h_{c_1} \wedge \dots \wedge h_{c_7} \right. \\ \left. \wedge \left(dA_{[3]} + \frac{1}{8} h_{c_1} h_{c_2} h_{c_3} h_{c_4} F^{c[4]} + t_{[1]}^{c[3]} h_{c_1} h_{c_2} h_{c_3} \right) F_{b[4]} + t_{[1] a[3]} de_{[9]}^{a[3]} \right], \quad (3.4.2)$$

that contains the additional independent fields $t_{[1] a[3]}$ and $e_{[9]}^{a[3]}$. The field equations derived from the parent action are given by

$$dA_{[3]} + \frac{1}{4} h_{c_1} h_{c_2} h_{c_3} h_{c_4} F^{c[4]} + t_{[1]}^{c[3]} h_{c_1} h_{c_2} h_{c_3} = 0, \quad (3.4.3)$$

$$d * (h^{a_1} \dots h^{a_4} F_{a[4]}) = 0, \quad (3.4.4)$$

$$dt_{[1]}^{a[3]} = 0, \quad (3.4.5)$$

$$de_{[9]}^{a[3]} + \frac{1}{7!} h^{a_1} h^{a_2} h^{a_3} F_{b[4]} \epsilon^{b[4]c[7]} h_{c_1} \wedge \dots \wedge h_{c_7} = 0. \quad (3.4.6)$$

The gauge symmetries of the action are

$$\delta A_{[3]} = d\lambda_{[2]} - h_{c_1} h_{c_2} h_{c_3} \psi_{[0]}^{c[3]}, \quad (3.4.7)$$

$$\delta F_{[0]}^{a[4]} = 0, \quad (3.4.8)$$

$$\delta t_{[1]}^{a[3]} = d\psi_{[0]}^{a[3]}, \quad (3.4.9)$$

$$\delta e_{[9]}^{a[3]} = d\xi_{[8]}^{a[3]}. \quad (3.4.10)$$

The equation (3.4.5) results from extremising with respect to $e_{[9]}^{a[3]}$. It implies that $t_{[1]}^{a[3]} = dC^{a[3]}$ and substituting this into the action we can absorb $C^{a[3]}$ into $A_{[3]}$ and the action becomes that of equation (3.4.1).

In order to descend on a different child action, we note that by using the Stückelberg-like gauge symmetry of $A_{[3]}$ with gauge parameter $\psi_{[0]}^{a[3]}$, one can completely gauge $A_{[3]}$ away, so that it disappears from the action (3.4.2). One can still perform differential gauge transformations with the two-form gauge parameter $\lambda_{[2]}$, but in order to stay in the gauge

where $A_{[3]}$ is zero, one has to compensate it with a residual transformation with parameter $\bar{\psi}^{a[3]} = \partial^{[a_1} \lambda^{a_2 a_3]}$.

We are thus left with a child action containing the fields, $e_{[9]}^{a[3]}$, $F_{a[4]}$ and $t_{[1]}^{a[3]}$, or better, containing only $e_{[9]}^{a[3]}$ and $t_{[1]}^{a[3]}$ as $F_{a[4]}$ can be expressed in terms of the totally antisymmetric part of $t_{[1]}^{a[3]}$ via (3.4.2). Note that under $\delta t_{[1]}^{a[3]} = d\psi_{[0]}^{c[3]}$ and in the gauge where $A_{[3]} = 0$, we have that $\psi^{a[3]} = \bar{\psi}^{a[3]} = \partial^{[a_1} \lambda^{a_2 a_3]}$ and therefore $\delta t_{[b|a_1 a_2 a_3]} = 0$, as it should. The field $t_{[1]}^{a[3]}$ plays the role of the connection $\omega_{[3]}^{a[10]}$ introduced in (3.3.1), upon dualisation of $t_{[1]}^{a[3]}$ on its form index and exchanging the role of form and frame indices. The equations of motion (3.4.5) imply that (i) $t_{[b|a_1 a_2 a_3]} = \partial_{[b} C_{a_1 a_2 a_3]}$, thereby re-introducing a three form on-shell, and (ii) the mixed-symmetric part $t_{a_1 a_2 a_3 || b} = 3 \partial_b \partial_{[a_1} \lambda_{a_2 a_3]}$. But this is precisely in the form of its residual gauge transformations in the gauge where $A_{[3]} = 0$, so that $t_{a[3] || b}$ is pure gauge and does not carry any local degree of freedom.

Let us demonstrate that the above action makes contact with the unfolded formalism given earlier in this section. Equation (3.4.6) can be written as

$$de_{[9]}^{a[3]} + h_{b_1} \wedge \dots \wedge h_{b_7} \wedge \tilde{\omega}_{[3]}^{a[3]b[7]} = 0. \quad (3.4.11)$$

where

$$\tilde{\omega}_{[3]}^{a[10]} = -\frac{1}{7!} h^{a_1} h^{a_2} h^{a_3} F_{c[4]} \epsilon^{a[7]c[4]} \quad (3.4.12)$$

plays the role of the connection appearing in (3.3.2). More precisely, it is the part of $\omega_{c[3]}^{a[3]}{}_{b[7]}$ that is antisymmetrised in its ten indices that are written as the two index blocks $c[3]$ and $b[7]$ that appears in (3.3.2), therefore, we rewrite

$$\tilde{\omega}_{c[3]}^{a[3]}{}_{c[7]} = \frac{3!7!}{10!} \epsilon_{c[7]f} F^{fa[3]}, \quad (3.4.13)$$

while performing the antisymmetrisation over the ten indices $a[10]$ on the right-hand side of (3.4.12) explicitly gives

$$\tilde{\omega}_{c[3]}^{a[10]} = \frac{4!}{10!} \epsilon^{a[10]b} F_{bc[3]}. \quad (3.4.14)$$

Note that the component of $e_{b[9]||a[3]}$ that transforms in the tensor product $[10] \otimes [2]$ of $GL(11)$ is pure gauge on-shell, as can be seen by suitably projecting (3.4.6) and using (3.4.10). Thus, it is only the $GL(11)$ -irreducible component $A_{b[9]||a[3]}$ of e that is glued to the zero-forms on-shell. To repeat, the components $e_{[b_1 \dots b_9 | b_{10}] a_1 a_2}$ are pure gauge on-shell and can therefore be eliminated in a gauge, leaving only the component $A_{b[9]||a[3]}$ with a differential gauge invariance in terms of the sole $GL(11)$ -irreducible component $\xi_{a[8]||b[3]}$: $\delta_\xi A_{b[9]||a[3]} = \partial_b \xi_{b[8]||a[3]}$. In this gauge, the field equations (3.4.6) then reduce to

$$\partial_a A_{a[9]||}{}^{b[3]} = \frac{4!7!}{10!} \epsilon_{a[10]c} F^{cb[3]}. \quad (3.4.15)$$

which we recognise as Equation (3.2.12) in the gauge where $\Xi^{a[9]||}_{b[2]}$ is set to zero. Upon acting with ∂^b and antisymmetrising over the four b indices, we get

$$\partial^b \partial_a A_{a[9]||}{}^{b[3]} = \frac{4!7!}{10!} \epsilon_{a[10]c} \partial^c \partial^{[b_1} A^{b_2 b_3 b_4]} , \quad (3.4.16)$$

which is nothing but the equation (3.2.1) up to an inessential coefficient.

3.5 Higher dualisations on-shell

In this section we will dualise the higher level components of the representation space \mathcal{T} given in equation (3.1.1). We consider a generic tensor in the list, say $F_{a[4]||b(n)}$, and dualise it on all n indices b so as to define

$$G^{(n)}_{c[10]||\dots||f[10]||a[4]} := \epsilon_{c[10]b_1} \epsilon_{d[10]b_2} \dots \epsilon_{f[10]b_n} F_{a[4]||}{}^{b(n)} . \quad (3.5.1)$$

Using the symmetries of $F_{a[4]||b(n)}$ we now show that the $G^{(n)}_{c[10]||\dots||f[10]||a[4]}$ is $GL(11)$ -irreducible. Explicitly, we find that

$$\begin{aligned} \epsilon^{c[10]f_1} G^{(n)}_{c[10]||\dots||f[10]||a[4]} &= (-10!) \delta_{b_1}^{f_1} \epsilon_{d[10]b_2} \dots \epsilon_{f_1 f[9]b_n} F_{a[4]||}{}^{b(n)} = \\ &= 10! \epsilon_{d[10]b_2} \dots \epsilon_{f[9]b_1 b_n} F_{a[4]||}{}^{b(n)} \equiv 0 , \end{aligned} \quad (3.5.2)$$

and

$$\begin{aligned} \epsilon^{c[10]a_1} G^{(n)}_{c[10]||\dots||f[10]||a[4]} &= (-10!) \delta_{b_1}^{a_1} \epsilon_{d[10]b_2} \dots \epsilon_{f[10]b_n} F_{a[4]||}{}^{b(n)} = \\ &= (-10!) \epsilon_{d[10]b_2} \dots \epsilon_{f[10]b_n} F_{b_1 a[3]||}{}^{b_1 b_2 \dots b(n)} \equiv 0 . \end{aligned} \quad (3.5.3)$$

We adopt the short-hand notation in which $G^{(n)}_{c[10]||\dots||f[10]||a[4]}$ is denoted by $G^{(n)}_{[10,\dots,10,4]}$, where the number of columns of height 10 is n . This object satisfies the trace conditions

$$\text{Tr}_{ij}^{10} G^{(n)}_{[10,\dots,10,4]} = 0 , \quad 1 \leq i < j \leq n , \quad (3.5.4)$$

$$\text{Tr}_{i n+1}^4 G^{(n)}_{[10,\dots,10,4]} = 0 , \quad 1 \leq i \leq n , \quad (3.5.5)$$

where we recall that the symbols Tr_{ij}^n means that one takes an n trace between indices in the i and j th column. To show the first relation we note that

$$\delta_{c_1}^{f_1} \dots \delta_{c_p}^{f_p} G^{(n)c[10]||\dots||f[10]||a[4]} = -p!(11-p)! \delta_{c[10-p]b_1}^{f[10-p]b_n} \epsilon_{d[10]b_2} \dots F_{a[4]||}{}^{b(n)} , \quad (3.5.6)$$

where $\delta_{d[p]}^{c[p]} = \delta^{[c_1]_{d_1} \dots \delta^{c_p]_{d_p}}$, so that the expression on the left-hand side vanishes only if the antisymmetrised product of Kronecker deltas on the right-hand side of the equation contains $\delta^{b_n}_{b_n}$, namely only when $p = 10$.

To show equation (3.5.5) we note that

$$G^{(n)a[p]c[10-p]||d[10]||\dots||f[10]||a[4]} = \epsilon^{a[p]c[10-p]b_1} \epsilon_{d[10]b_2} \dots \epsilon_{f[10]b_n} F_{a[4]||b_1}{}^{b_2 \dots b_n} , \quad (3.5.7)$$

which only gives zero when $p = 4$, i.e. when all the four indices a 's of $F_{a[4]||b_1}^{b_2 \dots b_n}$ are antisymmetrised with one of the n indices in the set $b(n)$. Obviously, the result is unchanged if one took four traces involving any another of the n columns of length 10 in $G_{[10, \dots, 10, 4]}^{(n)}$.

We now consider the derivatives acting on $G_{c[10]||\dots||f[10]||a[4]}^{(n)}$. In particular we observe that

$$\partial_{[a_1} G_{c[10]||d[10]||\dots||f[10]||a_2 \dots a_5]}^{(n)} = \epsilon^{c[10]b_1} \epsilon^{d[10]b_2} \dots \epsilon^{f[10]b_n} \partial_{[a_1} F_{a_2 \dots a_5]||}^{b(n)} = 0, \quad (3.5.8)$$

which is a consequence of the differential equations (3.1.10) obeyed by the hierarchy of tensors in the set \mathcal{T} in (3.1.1) together with the $GL(11)$ -irreducible symmetry properties (3.1.3) of these tensors. Whereupon using the generalised Poincare lemma [28] the $GL(11)$ -irreducible tensors $G_{[10, \dots, 10, 4]}^{(n)}$ can be expressed as generalised curvature tensors of $GL(11)$ -irreducible potentials:

$$G_{c[10]||\dots||f[10]||a[4]}^{(n)} = 4(10)^n \partial_c \dots \partial_f \partial_a A_{c[9]||\dots||f[9]||a[3]}^{(n)}, \quad (3.5.9)$$

The tensors $G^{(n)}(A^{(n)})$ are invariant under the following gauge transformations

$$\delta_\lambda A_{[9, \dots, 9, 3]}^{(n)} = d^{\{n\}} \lambda_{[9, \dots, 9, 8, 3]}^{(n)} + d^{\{n+1\}} \lambda_{[9, \dots, 9, 2]}^{(n+1)}. \quad (3.5.10)$$

where

$$\begin{aligned} d^{\{n\}} \lambda_{[9, \dots, 9, 8, 3]}^{(n)} &\rightarrow 9 \partial_{c_1^n} \Lambda_{c^1[9]||\dots||c^{n-1}[9]||c^n[8]||a[3]} + 9x \partial_{c_1^{n-1}} \Lambda_{c^1[9]||\dots||c^{n-1}[8]c_1^n||c^n[8]||a[3]} \\ &+ \dots + 9x \partial_{c_1^1} \Lambda_{c^1[8]c_1^n||\dots||c^{n-1}[9]||c^n[8]||a[3]}, \quad x = -\frac{9}{8}. \end{aligned} \quad (3.5.11)$$

The first term on the right-hand side of equation (3.4.10) can be depicted by the Young tableau

c_1	\dots	f_1	a_1
c_2	\dots	d_2	a_2
c_2	\dots	f_2	a_2
\vdots	\dots	\vdots	
\vdots	\dots	\vdots	
c_9	\dots	c_9	

(3.5.12)

Finally, the curvatures $G^{(n)}$, $n = 0, 1, \dots$ are related by

$$\partial_g G_{c[10]||\dots||f[10]||a[4]}^{(n)} = -\frac{1}{10!} \epsilon_g^{b[10]} G_{b[10]||c[10]||\dots||f[10]||a[4]}^{(n+1)} \quad (3.5.13)$$

leading to a corresponding duality relation for the first differentials of the various potentials:

$$\begin{aligned} \epsilon_{gb[10]} \partial^g A^{(n)}_{c[9]||\dots||f[9]||a[3]} &= 10 \partial_b A^{(n+1)}_{b[9]||c[9]||\dots||f[9]||a[3]} + 3 \partial_a \Lambda^{(1)}_{g[10],c[9]||\dots||f[9]||a[3]} \\ &+ Y(\partial_f \Lambda^{(2)}_{g[10],c[9]||\dots||f[8]||a[3]}) \end{aligned} \quad (3.5.14)$$

where the symbol Y means the projection of the $9n + 3$ indices $\{c[9]||\dots||f[9]||a[3]\}$ on the $GL(11)$ Young tableau with n columns of height 9 and one of height 3. Drawing from the experience we gained from the frame-like formulation of the gauge field $A_{[9,3]}$, we expect that the first-order duality relation (3.5.14) become free of inhomogeneous term when expressed in terms of the frame-like frame fields and connections.

4 Discussion

In this paper we have shown how the manifestly Lorentz and gauge covariant formulation of the irreducible representations of the Poincaré group leads naturally to a description of the dynamics of the massless point particle in terms of an infinite number of gauge fields which obey first order duality relations. Gauge fields of this type were automatically contained within the E_{11} non-linear realisation which is conjectured to be a symmetry of the underlying theory of strings and branes. The E_{11} symmetry acts on the infinite number of gauge fields rotating them into each other and this part of the symmetry can be thought of an extension of what we usually regard as a duality symmetry.

Duality symmetries have played an important part in theoretical physics and one may hope that the extension of the symmetry given in this paper may prove useful in future work. Certainly it will act as a very useful guide when formulating the equations of motion that follow from the E_{11} non-linear realisation.

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Appendix

In this appendix we give some of the notation used in this paper. While these definitions are given in the text it may not always be easy for the reader to find them and so we collect them here for easy reference. The first few sections of the paper are written without using elaborate notation so that the reader can get used to the subject, but as the paper progresses we need more and more indices and so we introduce a shorthand notation.

We separate blocks of antisymmetrised or symmetrised indices on the fields by putting a double bar, for example $A_{a_1 a_2 a_3 || b_1 \dots a_9}$. We eventually use a shorthand for blocks of antisymmetric and symmetric indices by denoting $A_{a[n]} \equiv A_{[a_1 \dots a_n]} \equiv A_{a_1 \dots a_n}$ for blocks

of antisymmetric indices and $S_{a(n)} \equiv S_{(a_1 \dots a_n)} \equiv S_{a_1 \dots a_n}$ for symmetrised indices. We use the strength-one (anti)symmetrisation convention.

In the early sections of the paper we denote antisymmetrisation in the usual way that is $F_{a_1 a_2 a_3 a_4} = 4\partial_{[a_1} A_{a_2 a_3 a_4]}$. However, once we have more indices to cope with we adopt the convention that when an index with the same Latin label occurs in the same up, or down, position in an equation, it is automatically antisymmetrised. For example, when we write $F_{a[4]} = 4\partial_a A_{a[3]}$ we automatically mean $F_{a_1 a_2 a_3 a_4} = 4\partial_{[a_1} A_{a_2 a_3 a_4]}$.

When we are discussing forms we label the degree of the form by a number in square brackets written as a subscript, for example $A_{[3]} = \frac{1}{3!} dx^{a_1} \wedge dx^{a_2} \wedge dx^{a_3} A_{a_1 a_2 a_3}$.

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